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Notes on Ponomarev-systems

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Abstract Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system. We prove that f is a sequence-covering (resp., pseudo-sequence-covering, subsequence-covering) map iff $\mathcal{P}$ is a strong cs-network (resp., strong cs^*-network, pseudo-strong cs^*-network) for X. Moreover, "subsequence-covering" can be replaced by "sequentially-quotient", and "pseudo-strong cs^*-network" can be replaced by "pseudo-strong cs-network". As applications of these results, we get many well-known results on images of metric spaces and more.

Keywords compact map, s-map, cs-map, strong s-map, sequence-covering map, pseudo-sequence-covering map, sequentially-quotient map, subsequence-covering map, Ponomarev-system

§1. Introduction

Spaces with point-countable cfp-networks (resp., point-countable cs-networks, point-countable cs^*-networks) can be characterized as s-images of a metric space M under a covering-map f, and many results have been proved ([1], [10], [11], [14]). Recently, some authors have tried to generalize these results ([5], [12], [15]). In [5], Y. Ge and J. Shen have obtained the following.

Theorem 1.1. ([5], Theorem 2.1) Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system, then f is a compact-covering map iff $\mathcal{P}$ is a strong k-network for X.

Take Theorem 1.1 into account, and note that compact-covering maps, sequence-covering maps, pseudo-sequence-covering maps, subsequence-covering maps and sequentially-quotient maps have a closed relation ([3], [4], [14]), the following question naturally arises.

Question 1.2. Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system. What is the necessary and sufficient condition such that f is a sequence-covering (pseudo-sequence-covering, subsequence-covering, sequentially-quotient) map?

In this paper, we introduce definitions of strong cs-network, strong cs^*-network, pseudo-strong cs-network and pseudo-strong cs^*-network as modifications of strong k-network in [5] to give necessary and sufficient conditions such that f is sequence-covering (resp., pseudo-sequence-covering, subsequence-covering, sequentially-quotient) where $(f, M, X, \mathcal{P})$ is a Ponomarev-system. As applications of these results, we get many well-known results on images of metric spaces and more.

Throughout this paper, all spaces are assumed to be regular and T_1, all maps are assumed continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers, ω denotes $\mathbb{N} \cup \{0\}$ and convergent

sequence includes its limit point. Let $f : X \to Y$ be a map and $\mathcal{P}$ be a collection of subsets of X, we denote $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$ and $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$.

Definition 1.3. Let A be a subset of a space X and $\mathcal{P}$ be a collection of subsets of X.

(1) $\mathcal{P}$ is a network at x in X, if $x \in P$ for every $P \in \mathcal{P}$ and whenever $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.

(2) $\mathcal{P}$ is a network for X, if whenever $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.

(3) $\mathcal{P}$ is a k-network for X, if whenever $K \subset U$ with K compact and U open in X, then $K \subset \bigcup \mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{P}$.

(4) $\mathcal{P}$ is a cfp-cover for A in X, if $\mathcal{P}$ is a cover for A in X such that it can be precisely refined by some finite cover consisting of closed subsets of A.

(5) $\mathcal{P}$ is a cfp-network for A in X, if whenever K is a compact subset of A and $K \subset U$ with U open in X, there exists a subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $\mathcal{F}$ is a cfp-cover for K in A and $\bigcup \mathcal{F} \subset U$. A such $\mathcal{P}$ in [5] is called to have property cc for A.

(6) $\mathcal{P}$ is a strong k-network for X, if whenever K is a compact subset of X, there exists a countable subfamily $\mathcal{P}_K$ of $\mathcal{P}$ such that $\mathcal{P}_K$ is a cfp-network for K in X. Note that there is a different definition of strong k-network in [1].

(7) $\mathcal{P}$ is a cs-network for A in X, if whenever S is a convergent sequence in A converging to $x \in A \cap U$ with U open in X, then S is eventually in $P \subset U$ for some $P \in \mathcal{P}$.

(8) $\mathcal{P}$ is a cs^*-network for A in X, if whenever S is a convergent sequence converging to $x \in A \cap U$ with U open in X, then S is frequently in $P \subset U$ for some $P \in \mathcal{P}$.

(9) $\mathcal{P}$ is compact-countable, if $\{P \in \mathcal{P} : K \cap P \neq \emptyset\}$ is countable for each compact subset K of X.

(10) $\mathcal{P}$ is point-finite, if $\{P \in \mathcal{P} : x \in P\}$ is finite for each point $x \in X$.

(11) $\mathcal{P}$ is locally finite, if for each point $x \in X$, there is a neighborhood U of X such that U meets only finitely many members of $\mathcal{P}$.

(12) $\mathcal{P}$ is σ-locally finite, if $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ where each $\mathcal{P}_n$ is a locally finite collection.

(13) $\mathcal{P}$ is point-countable, if $\{P \in \mathcal{P} : x \in P\}$ is countable for each point $x \in X$. Note that every point-finite (locally countable, compact-countable) collection is a point-countable collection.

(14) If $A = X$, then a cs-network (resp., cs^*-network, cfp-cover, cfp-network) for A in X is called a cs-network (resp., cs^*-network, cfp-cover, cfp-network) for X (see [5], [12]).

Definition 1.4. Let $f : X \to Y$ be a map.

(1) f is a compact-covering map, if each compact subset of Y is the image of some compact subset of X.

(2) f is an s-map, if whenever $y \in Y$, then $f^{-1}(y)$ is a separable subset of X.

(3) f is a strong s-map, if for each $y \in Y$, there exists a neighborhood V of y in Y such that $f^{-1}(V)$ is a separable subset of X.

(4) f is a cs-map, if whenever K is a compact subset of Y, then $f^{-1}(K)$ is a separable subset of X.

(5) f is a compact map, if whenever $y \in Y$, then $f^{-1}(y)$ is a compact subset of X.

(6) f is a σ-map, if there is a base $\mathcal{B}$ for X such that $f(\mathcal{B})$ is a σ-locally finite collection of subsets of Y.

(7) f is a sequence-covering map, if each convergent sequence in Y is the image of some convergent sequence in X.

(8) f is a pseudo-sequence-covering map, if each convergent sequence in Y is the image of some compact subset of X.

(9) f is a sequentially-quotient map, if for each convergent sequence S in Y, there is a convergent sequence L in X such that $f(L)$ is a subsequence of S.

(10) f is a subsequence-covering map, if for each convergent sequence S in Y, there is a compact subset K of X such that $f(K)$ is a subsequence of S.

Definition 1.5. [14] Let X be a space.

(1) X is an $\aleph_0$-space, if X has a countable cs-network. Note that "cs-" can be replaced by "k-" or "cs^*-".

(2) X is an $\aleph$-space, if X has a σ-locally finite cs-network. Note that "cs-" can be replaced by "k-" or "cs^*-".

(3) X is a Fréchet space, if whenever $x \in \overline{A}$ with $A \subset X$, then there is a sequence in A converging to x.

(4) X is a sequential space, if whenever A is a non closed subset of X, then there is a sequence in A converging to a point not in A.

Definition 1.6. [5] Let $\mathcal{P}$ be a network for a space X. Assume that $\mathcal{P}$ is closed under finite intersections, and there exists a countable subfamily $\mathcal{P}_x$ of $\mathcal{P}$ such that $\mathcal{P}_x$ is a network at x in X for every $x \in X$. Put $\mathcal{P} = \{P_\beta : \beta \in \Gamma\}$. For every $n \in \mathbb{N}$, put $\Gamma_n = \Gamma$ and endowed Γ_n with the discrete topology. Put

$$M = \Big\{b = (\beta_n) \in \prod_{n\in\mathbb{N}} \Gamma_n : \{P_{\beta_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x_b \in X\Big\}.$$

Then M, which is a subspace of the product space $\prod_{n\in\mathbb{N}} \Gamma_n$, is a metric space and x_b is unique for every $b \in M$. Moreover $x_b = \bigcap_{n\in\mathbb{N}} P_{\beta_n}$. Define $f : M \to X$ by $f(b) = x_b$, then f is a map and $(f, M, X, \mathcal{P})$ is called a Ponomarev-system.

Remark 1.7. If $\mathcal{P}$ is a point-countable network for X, then $\mathcal{P}_x = \{P \in \mathcal{P} : x \in P\} \subset \mathcal{P}$ is a countable network at x in X for every $x \in X$. It implies that the Ponomarev-system $(f, M, X, \mathcal{P})$ exists.

For terms which are not defined here, please refer to [2] and [14]

§2. Results

In the following, $(f, M, X, \mathcal{P})$ denotes a Ponomarev-system where f and M are defined in Definition 1.6.

Firstly we introduce some definitions.

Definition 2.1. Let $\mathcal{P}$ be a cs-network for a convergent sequence S in X where $S \subset U$ with U open in X, then a subfamily $\mathcal{F}$ of $\mathcal{P}$ has property $cs(S, U)$ if it satisfies the following.

(1) $\mathcal{F}$ is finite,

(2) $\emptyset \neq F \cap S \subset F \subset U$ for every $F \in \mathcal{F}$,

(3) If $x \in S$, then there is a unique $F \in \mathcal{F}$ such that $x \in F$,

(4) If $F \in \mathcal{F}$ contains the limit point of S, then $S - F$ is finite.

The following lemma proves that a family having property $cs(S, U)$ exists.

Lemma 2.2. If $\mathcal{P}$ is a cs-network for a convergent sequence S in X where $S \subset U$ with U open in X, then there is a subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $\mathcal{F}$ has property $cs(S, U)$.

Proof. Let $S = \{x_m : m \in \omega\}$ converging to $x_0 \in X$. Since $\mathcal{P}$ is a cs-network for S in X, there is $P_0 \in \mathcal{P}$ such that S is eventually in P_0 and $P_0 \subset U$. Because $S - U$ is finite, $S - U = \{x_{m_i} : i = 1, \cdots, k\}$ for some $k \in \mathbb{N}$. For every $i \in \{1, \cdots, k\}$, note that $U - \big(S - \{x_{m_i}\}\big)$ is an open neighborhood of x_{m_i} in X, so there is $P_i \in \mathcal{P}$ such that $x_{m_i} \in P_i \subset U - \big(S - \{x_{m_i}\}\big)$. Put $\mathcal{F} = \{P_i : i = 0, \cdots, k\}$, then $\mathcal{F}$ satisfies required conditions.

Definition 2.3. Let $\mathcal{P}$ be a cs^*-network for a convergent sequence S in X where $S \subset U$ with U open in X, then a subfamily $\mathcal{F}$ of $\mathcal{P}$ has property $cs^*(S, U)$ if it satisfies the following.

(1) $\mathcal{F}$ is finite,

(2) $\emptyset \neq F \cap S \subset F \subset U$ for every $F \in \mathcal{F}$,

(3) If $x \in S$, then there is some $F \in \mathcal{F}$ such that $x \in F$,

(4) $F \cap S$ is closed for every $F \in \mathcal{F}$.

The following lemma proves that a family having property $cs^*(S, U)$, if $\mathcal{P}$ is point-countable.

Lemma 2.4. If $\mathcal{P}$ is a point-countable cs^*-network for a convergent sequence S in X where $S \subset U$ with U open in X, then there is a subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $\mathcal{F}$ has property $cs^*(S, U)$.

Proof. Let $S = \{x_n : n \in \omega\}$ converging to $x_0 \in X$. Since $\mathcal{P}$ is point-countable cs^*-network for S in X, $\{P \in \mathcal{P} : x_0 \in P \subset U\}$ is non-empty and countable. Put $\{P \in \mathcal{P} : x_0 \in P \subset U\} = \{P_i : i \in \mathbb{N}\}$. We shall show that there exists an $n_0 \in \mathbb{N}$ such that $x_n \in \bigcup_{i=1}^{n_0} P_i$ for all but finitely many $n \in \mathbb{N}$. If not, we can choose a subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$ as follows.

$$x_{n_1} \in (S - P_1) \cap P_{n_1} \text{ for some } P_{n_1} \in \mathcal{P},$$

$$x_{n_k} \in (S - \bigcup_{i=1}^{n_{k-1}} P_i) \cap P_{n_k} \text{ for some } P_{n_k} \in \mathcal{P}, \text{ for all } k > 1.$$

So each P_i only includes finitely many elements of $\{x_{n_k} : k \in \mathbb{N}\}$. But $\mathcal{P}$ is a cs^*-network for S in X, then there is $P \in \mathcal{P}$ such that $\{x_0\} \cup \{x_{n_k} : k \in \mathbb{N}\}$ is frequently in P. Thus $P = P_m$ for some $m \in \mathbb{N}$. Hence P_m includes infinitely many elements of $\{x_{n_k} : k \in \mathbb{N}\}$, a contradiction. Then we can put $S - \bigcup_{i=1}^{n_0} P_i = \{x_{n_i} : i = 1, \cdots, k\}$ for some $k \in \mathbb{N}$. For each $i \in \{1, \cdots, k\}$ there is $F_i \in \mathcal{P}$ such that $x_{n_i} \in F_i \subset \big(U - (S - \{x_{n_i}\})\big)$. Put $\mathcal{F} = \{F_i : i = 1, \cdots, k\} \cup \{P_i : i = 1, \cdots, n_0\}$, then $\mathcal{F}$ satisfies required conditions.

The following notions are modifications of the strong k-network in [5].

Definition 2.5. Let $\mathcal{P}$ be a collection of subsets of a space X.

(1) $\mathcal{P}$ is a strong cs-network for X, if whenever S is a convergent sequence in X, then there exists a countable subfamily $\mathcal{P}_S$ of $\mathcal{P}$ such that $\mathcal{P}_S$ is a cs-network for S in X.

(2) $\mathcal{P}$ is a strong cs^*-network for X, if whenever S is a convergent sequence in X, then there exists a countable subfamily $\mathcal{P}_S$ of $\mathcal{P}$ such that $\mathcal{P}_S$ is a cs^*-network for S in X.

(3) $\mathcal{P}$ is a pseudo-strong cs-network for X, if whenever S is a convergent sequence in X, then there exists a countable subfamily $\mathcal{P}_S$ of $\mathcal{P}$ such that $\mathcal{P}_S$ is a cs-network in X for some convergent subsequence T of S.

(4) $\mathcal{P}$ is a pseudo-strong cs^*-network for X, if whenever S is a convergent sequence in X, then there exists a countable subfamily $\mathcal{P}_S$ of $\mathcal{P}$ such that $\mathcal{P}_S$ is a cs^*-network in X for some convergent subsequence T of S.

Remark 2.6. We have following implications from the above definitions.

(1) Strong cs-network $\Rightarrow$ cs-network.

(2) Strong cs-network $\Rightarrow$ strong cs^*-network (pseudo-strong cs-network) $\Rightarrow$ pseudo-strong cs^*-network $\Rightarrow$ cs^*-network.

The following example proves that some inverse implications in Remark 2.6 do not hold.

Example 2.7. There exists a Ponomarev-system $(f, M, X, \mathcal{P})$ such that $\mathcal{P}$ is a cs-network for X, but $\mathcal{P}$ is not a strong cs-network for X.

Proof. Let X be the sequential fan space S_ω [8], then X has not any countable base at x_0, where x_0 is the non-isolated point in X. Put $\mathcal{P} = \{U \subset X : U \text{ is open in } X\} \cup \big\{\{x_0\}\big\}$, then $\mathcal{P}$ is a network for X, and there exists a countable subfamily $\mathcal{P}_x$ of $\mathcal{P}$ such that $\mathcal{P}_x$ is a network at x in X for every $x \in X$. Therefore the Ponomarev-system $(f, M, X, \mathcal{P})$ exists.

Since $\mathcal{P}$ contains a base of X, $\mathcal{P}$ is a cs-network for X. We shall prove that $\mathcal{P}$ is not a strong cs-network for X. Let S be a non-trivial convergent sequence converging to x_0. If $\mathcal{P}$ is a strong cs-network for X, there exists a countable subfamily $\mathcal{P}_S$ of $\mathcal{P}$ such that $\mathcal{P}_S$ is a cs-network for S in X. Note that every element of $\mathcal{P}_S - \big\{\{x_0\}\big\}$ is open in X and $\mathcal{P}_S - \big\{\{x_0\}\big\}$ is a countable network at x_0 in X. Therefore $\mathcal{P}_S - \big\{\{x_0\}\big\}$ is a countable neighborhood base at x_0 in X. This contradicts that X has not any countable neighborhood base at x_0. It implies that $\mathcal{P}$ is not a strong cs-network for X.

Remark 2.8. Similarly, we get that there exists a Ponomarev-system $(f, M, X, \mathcal{P})$ such that $\mathcal{P}$ is a cs^*-network for X, but $\mathcal{P}$ is not a pseudo-strong cs^*-network for X where X and $\mathcal{P}$ are the same in example 2.7. Moreover $\mathcal{P}$ is not neither a pseudo-strong cs-network for X nor a strong cs^*-network for X by remark 2.6. But we don't know whether remaining inverse implications in remark 2.6 do not hold.

The following lemma establishes a equivalent condition between a strong cs-network and a cs-network.

Lemma 2.9. If $\mathcal{P}$ is a point-countable family consisting of subsets of a space X, then the following are equivalent.

(1) $\mathcal{P}$ is a strong cs-network for X,

(2) $\mathcal{P}$ is a cs-network for X.

Proof. (1) $\Rightarrow$ (2). By Remark 2.6.

(2) $\Rightarrow$ (1). Let S be a convergent sequence in X. Since $\mathcal{P}$ is point-countable, then $\mathcal{P}_S = \{P \in \mathcal{P} : P \cap S \neq \emptyset\}$ is countable. Obviously $\mathcal{P}_S$ is a cs-network for S in X. It implies that $\mathcal{P}$ is a strong cs-network for X.

Moreover we get that.

Lemma 2.10. Let $\mathcal{P}$ be a point-countable cs-network for X. If each compact subset of X is first-countable, then $\mathcal{P}$ is a cfp-network for X.

Proof. Let K be a compact subset of X and $K \subset U$ with U open in X. It follows from Lemma 1.2 in [9] that there are $P_x, Q_x \in \mathcal{P}$ such that $x \in \mathrm{int}_K(Q_x \cap K) \subset G_x \subset \mathrm{cl}_K G_x \subset \mathrm{int}_K(P_x \cap K) \subset P_x \subset U$ with some G_x open in K for every $x \in K$. Since K is compact, there is a finite subset F of K such that $\{G_x : x \in F\}$ covers K. Then $\mathcal{F} = \{P_x : x \in F\}$ is a cfp-cover for K in X with $\bigcup \mathcal{F} \subset U$. It implies that $\mathcal{P}$ is a cfp-network for X.

We also have a equivalent condition for a strong cs^*-network, pseudo-strong cs^*-network and a cs^*-network as follows.

Lemma 2.11. If a family $\mathcal{P}$ is a point-countable family consisting of subsets of a space X, then the following are equivalent.

(1) $\mathcal{P}$ is a strong cs^*-network for X ,

(2) $\mathcal{P}$ is a pseudo-strong cs^*-network for X,

(3) $\mathcal{P}$ is a cs^*-network for X.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). By Remark 2.6.

(3) $\Rightarrow$ (1). Let S be a convergent sequence in X. Since $\mathcal{P}$ is point-countable, then $\mathcal{P}_S = \{P \in \mathcal{P} : P \cap S \neq \emptyset\}$ is countable. Obviously $\mathcal{P}_S$ is a cs^*-network for S in X. It implies that $\mathcal{P}$ is a strong cs^*-network for X.

Regarding the relations between covering-maps we have the following.

Lemma 2.12. {[3], Remark 1.8} Let $f : X \to Y$ be a map.

(1) If f is quotient and X is sequential, then f is sequentially-quotient.

(2) If Y is sequential and f is sequentially-quotient, then f is quotient.

Lemma 2.13. {[4], Remark 5} Let $f : X \to Y$ be a map.

(1) If f is compact-covering or sequence-covering, then f is pseudo-sequence-covering.

(2) If f is pseudo-sequence-covering or sequentially-quotient, then f is subsequence-covering.

Moreover we get that.

Lemma 2.14. Let $f : X \to Y$ be a map. If X is sequential and f is subsequence-covering, then f is sequentially-quotient.

Proof. Let S be a convergent sequence converging to a point $y \in Y$. Since f is subsequence-covering, there is a compact subset K in X such that $f(K)$ is a convergent subsequence of S. Put $f(K) = \{y\} \cup \{y_n : n \in \mathbb{N}\}$ where $\{y_n : n \in \mathbb{N}\}$ converges to y. For each $n \in \mathbb{N}$ pick $x_n \in f^{-1}(y_n) \cap K$, then $\{x_n : n \in \mathbb{N}\} \subset K$. Note that K is a compact subset in a sequential space, K is sequentially compact. So there is a convergent subsequence $\{x\} \cup \{x_{n_k} : k \in \mathbb{N}\}$ of $\{x\} \cup \{x_n : n \in \mathbb{N}\}$ that converges to $x \in f^{-1}(y)$. Then $\{y\} \cup \{f(x_{n_k}) : k \in \mathbb{N}\}$ is a convergent subsequence of $\{y\} \cup \{y_n : n \in \mathbb{N}\}$. Therefore $\{y\} \cup \{f(x_{n_k}) : k \in \mathbb{N}\}$ is a convergent subsequence of S. This proves that f is sequentially-quotient.

Now we establish a relation between covers and map in a ponomarevsystem.

Lemma 2.15. Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system, then the following hold.

(1) f is a compact map iff $\mathcal{P}$ is point-finite.

(2) f is an s-map iff $\mathcal{P}$ is point-countable.

(3) f is a cs-map iff $\mathcal{P}$ is compact-countable.

(4) f is a strong s-map iff $\mathcal{P}$ is locally countable.

Proof. (1) and (2). By Proposition 2.1 in [5].

(3) Necessary. Conversely, if $\mathcal{P}$ is not compact-countable, then there exists some compact subset K of X such that $\Lambda = \{\beta \in \Gamma : P_\beta \cap K \neq \emptyset\}$ is uncountable. Let $x \in K$ and $\{P_{\beta_n} : n \in \mathbb{N}\}$ is a network at x in X. For every $\beta \in \Lambda$ put $c_\beta = (\gamma_n)$ where $\gamma_1 = \beta$ and $\gamma_n = \beta_{n-1}$ for $n > 1$. Then $\{P_{\gamma_n} : n \in \mathbb{N}\}$ is a network at x, so $c_\beta \in f^{-1}(K)$. Put $U_\beta = \{c = (\gamma_n) \in M : \gamma_1 = \beta\}$ for every $\beta \in \Lambda$. We shall prove that $\{U_\beta : \beta \in \Lambda\}$ is an open cover for $f^{-1}(K)$ in M. Note that every U_β is open and non-empty, and if $c = (\gamma_n) \in f^{-1}(K)$, then $\{P_{\gamma_n} : n \in \mathbb{N}\} \subset \mathcal{P}$ is a network at $f(c) \in K$. It implies that $P_{\gamma_1} \cap K \neq \emptyset$. Then $\gamma_1 = \beta$ for some $\beta \in \Lambda$. Hence $\{U_\beta : \beta \in \Lambda\}$ is an open cover for $f^{-1}(K)$ in M. Since f is a cs-map, $f^{-1}(K)$ is separable in M. Then $\{U_\beta : \beta \in \Lambda\}$ has a countable subcover. It is a contradiction because $U_\beta \cap U_\gamma = \emptyset$ whenever $\beta \neq \gamma$.

Sufficiency. Let K be a compact subset of X. Since $\mathcal{P}$ is compact-countable, $\Lambda_n = \{\beta \in \Gamma_n : P_\beta \cap K \neq \emptyset\}$ is countable for every $n \in \mathbb{N}$. Then $f^{-1}(K) \subset \prod_{n\in\mathbb{N}} \Lambda_n$. Since $\prod_{n\in\mathbb{N}} \Lambda_n$ is a hereditarily separable space, it implies that $f^{-1}(K)$ is separable in M, i.e, f is a cs-map.

(4) It is similar to the proof of (3).

From Theorem 2.1 in [5] (see Theorem 1.1) the authors have obtained the following.

Corollary 2.16. {[5], Proposition 3.1} Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system, then the following are equivalent.

(1) f is a compact-covering s-map,

(2) $\mathcal{P}$ is a point-countable strong k-network,

(3) $\mathcal{P}$ is a point-countable cfp-network.

Note that "s-" and "point-countable" in Corollary 2.16 can be replaced by "compact" and "point-finite" respectively ([5], Remark 3.1.) Using Lemma 2.15 we get that "s-" and "point-countable" in Corollary 2.16 can be also replaced by "cs-" and "compact-countable", or "strong s-" and "locally countable" respectively. Moreover, the following holds.

Corollary 2.17. {[12], Corollary 8} The following are equivalent for a space X.

(1) X is a compact-covering s-image of a metric space,

(2) X has a point-countable cfp-network.

Proof. (1) $\Rightarrow$ (2). Let $f : M \to X$ be a compact-covering s-map from a metric space M onto X. Since M is metric, M has a σ-locally finite base $\mathcal{B}$. Then $f(\mathcal{B})$ is a point-countable cfp-network for X.

(2) $\Rightarrow$ (1). Let $\mathcal{P}$ be a point-countable cfp-network for X. It follows from Remark 1.7 that the Ponomarev-system $(f, M, X, \mathcal{P})$ exists. Then f is a compact-covering s-map by Corollary 2.16. It implies that X is a compact-covering s-image of a metric space.

Next we give a technical lemma which plays an important role in the following parts.

Lemma 2.18. Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system, $b = (\beta_n) \in M$ where $\{P_{\beta_n} : n \in \mathbb{N}\}$ is a network at some $x_b \in X$ and

$$U_n = \{c = (\gamma_i) \in M : \gamma_i = \beta_i \text{ for all } i \leqslant n\},$$

for every $n \in \mathbb{N}$. Then we have.

(1) $\{U_n : n \in \mathbb{N}\}$ is a base at b in M.

(2) $f(U_n) = \bigcap_{i=1}^{n} P_{\beta_i}$ for every $n \in \mathbb{N}$.

Proof. (1). By definition of the product topology of a countable family consisting of discrete spaces.

(2). For each $n \in \mathbb{N}$, let $x \in f(U_n)$. Then $x = f(c)$ for some $c = (\gamma_i) \in U_n$. It implies that $x = \bigcap_{i\in\mathbb{N}} P_{\gamma_i} \subset \bigcap_{i=1}^{n} P_{\gamma_i} = \bigcap_{i=1}^{n} P_{\beta_i}$, i.e, $f(U_n) \subset \bigcap_{i=1}^{n} P_{\beta_i}$.

Conversely, let $x \in \bigcap_{i=1}^{n} P_{\beta_i}$. Then $x = f(c)$ with $c = (\gamma_i) \in M$. Note that for each $i \in \mathbb{N}$ there exists some $\theta_{n+i} \in \Gamma_{n+i}$ such that $\theta_{n+i} = \gamma_i$. Put $d = (\theta_i)$ with $\theta_i = \beta_i$ for all $i \leqslant n$. Then we get $d \in U_n$ and $f(d) = x$. It implies that $\bigcap_{i=1}^{n} P_{\beta_i} \subset f(U_n)$.

By the above we get $f(U_n) = \bigcap_{i=1}^{n} P_{\beta_i}$.

Now we give a necessary and sufficient condition such that f is a sequence-covering map.

Theorem 2.19. Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system, then the following are equivalent.

(1) f is a sequence-covering map,

(2) $\mathcal{P}$ is a strong cs-network for X.

Proof. (1) $\Rightarrow$(2). Let S be a convergent sequence in X, we shall prove that there is a countable subfamily $\mathcal{P}_S$ of $\mathcal{P}$ such that $\mathcal{P}_S$ is a cs-network for S in X. Indeed, since f is sequence-covering, there exists a convergent sequence C in M such that $S = f(C)$. Put $\Lambda = \bigcup\{p_n(C) : n \in \mathbb{N}\}$, where every $p_n : \prod_{i\in\mathbb{N}} \Gamma_i \to \Gamma_n$ is a projection, then Λ is countable. Let

$$\mathcal{P}_S = \Big\{ \bigcap_{\beta\in\Delta} P_\beta : \Delta \subset \Lambda, \Delta \text{ is finite}\Big\}.$$

Since $\mathcal{P}$ is closed under finite intersections, $\mathcal{P}_S$ is a countable subfamily of $\mathcal{P}$. It suffices to show that $\mathcal{P}_S$ is a cs-network for S in X. Let L be a convergent sequence in S converging to $x \in U$ with U open in X. Since f is sequence-covering, there exists a convergent subsequence T of C in M such that $f(T) = L$. We get that T converges to some $b \in f^{-1}(x) \subset f^{-1}(U)$. Let $b = (\beta_n)$. For each $n \in \mathbb{N}$ put

$$U_n = \{c = (\gamma_i) \in M : \gamma_i = \beta_i \text{ for all } i \leqslant n\}.$$

It follows from Lemma 2.18 that $\{U_n : n \in \mathbb{N}\}$ is a base at b in M. Since T converges to $b \in f^{-1}(U)$ which is open in M, T is eventually in some U_n with $U_n \subset f^{-1}(U)$. Therefore L is eventually in $f(U_n) \subset U$. Since $f(U_n) = \bigcap_{i=1}^{n} P_{\beta_i}$ by Lemma 2.18 and $\bigcap_{i=1}^{n} P_{\beta_i}$ is an element of $\mathcal{P}_S$, we get that $\mathcal{P}_S$ is a cs-network for S in X.

(2) $\Rightarrow$ (1). Let $S = \{x_m : m \in \omega\}$ be a convergent sequence in X converging to x_0. We shall prove that $S = f(C)$ for some convergent sequence C in M. Assume that all x_m's are distinct. Since $\mathcal{P}$ is a strong cs-network for X, there exists a countable subfamily $\mathcal{P}_S$ of $\mathcal{P}$ such that $\mathcal{P}_S$ is a cs-network for S. It follows from Lemma 2.2 that there exists a subfamily $\mathcal{F}$ of $\mathcal{P}_S$

such that $\mathcal{F}$ has property $cs(S, X)$. Since $\mathcal{P}_S$ is countable, $\{\mathcal{F} \subset \mathcal{P}_S : \mathcal{F}$ has property $cs(S, X)\}$ is countable by finiteness of $\mathcal{F}$. Put

$$\{\mathcal{F} \subset \mathcal{P} : \mathcal{F} \text{ has property } cs(S, X)\} = \{\mathcal{F}_n : n \in \mathbb{N}\},$$

and put $\mathcal{F}_n = \{P_\beta : \beta \in \Delta_n\}$ for every $n \in \mathbb{N}$ where Δ_n is a finite subset of Γ_n. For every $m \in \omega$ and $n \in \mathbb{N}$, since $\mathcal{F}_n$ has property $cs(S, X)$, there is a unique $\beta_{n,m} \in \Delta_n$ such that $x_m \in P_{\beta_{n,m}} \in \mathcal{F}_n$. Put $b_m = (\beta_{n,m}) \in \prod_{n\in\mathbb{N}} \Delta_n$ and $C = \{b_m : m \in \omega\}$, we shall prove that C is a convergent sequence in M and $f(C) = S$.

To show $C \subset M$ and $f(C) = S$ it suffices to prove that $\{P_{\beta_{n,m}} : n \in \mathbb{N}\}$ is a network in X at x_m for every $m \in \omega$. Indeed, let U be an open neighborhood of x_m in X. We consider two following cases.

(a) $x_m = x_0$.

We get that $U \cap S$ is a convergent sequence in X and $(S \cap U) \subset U$. It follows from Lemma 2.2 that there is a subfamily $\mathcal{F}$ of $\mathcal{P}_S$ such that $\mathcal{F}$ to have property $cs(S \cap U, U)$. Since $S - (S \cap U)$ is finite, put $S - (S \cap U) = \{x_{m_i} : i = 1, \cdots, l\}$. For every $i \in \{1, \dots, l\}$, note that $X - (S - \{x_{m_i}\})$ is an open neighborhood of x_{m_i} in X, so there is $P_i \in \mathcal{P}_S$ such that $x_{m_i} \in P_i \subset X - (S - \{x_{m_i}\})$. It is easy to see that $\mathcal{F} \cup \{P_i : i = 1, \cdots, l\}$ has property $cs(S, X)$. So there is $k \in \mathbb{N}$ such that $\mathcal{F} \cup \{P_i : i = 1, \cdots, l\} = \mathcal{F}_k$. Thus $x_0 \in P_{\beta_{k,0}} \in \mathcal{F}_k$. Note that $P_{\beta_{k,0}}$ must be an element of $\mathcal{F}$ which has property $cs(S \cap U, U)$. It implies that $x_0 \in P_{\beta_{k,0}} \subset U$.

(b) $x_m \neq x_0$.

We get that $S - \{x_m\}$ is a convergent sequence in X and $S - \{x_m\} \subset X - \{x_m\}$ with $X - \{x_m\}$ open. It follows from Lemma 2.2 that there exists a subfamily $\mathcal{F}$ of $\mathcal{P}_S$ such that $\mathcal{F}$ has property $cs(S - \{x_m\}, X - \{x_m\})$. Note that $U - (S - \{x_m\})$ is an open neighborhood of x_m, so there exists $P_m \in \mathcal{P}_S$ such that $x_m \in P_m \subset U - (S - \{x_m\})$. It is easy to see that $\mathcal{F} \cup \{P_m\}$ has property $cs(S, X)$. Hence there is $k \in \mathbb{N}$ such that $\mathcal{F} \cup \{P_m\} = \mathcal{F}_k$, then $x_m \in P_{\beta_{k,m}} = P_m \subset U$.

By the above we get that $x_m \in P_{\beta_{k,m}} \subset U$ for every $m \in \omega$. Then $\{P_{\beta_{n,m}} : n \in \mathbb{N}\}$ is a network in X at x_m for every $m \in \omega$. It implies that $C \subset M$ and $f(C) = S$. To complete the proof we shall prove that $C = \{b_m : m \in \omega\}$ converging to b_0. For every $k \in \mathbb{N}$ there is a unique $\beta_{k,0} \in \Delta_k$ such that $x_0 \in P_{\beta_{k,0}} \in \mathcal{F}_k$. Since $\mathcal{F}_k$ has property $cs(S, X)$, $S - P_{\beta_{k,0}}$ is finite. So there is $m_k \in \mathbb{N}$ such that $x_m \in P_{\beta_{k,0}}$ for every $m > m_k$. Note that $x_m \in P_{\beta_{k,m}} \in \mathcal{F}_k$. Thus $\beta_{k,m} = \beta_{k,0}$ for every $m > m_k$. So $\{\beta_{k,m} : m \in \omega\}$ converging to $\beta_{k,0}$ as $m \to \infty$. Hence $C = \{b_m : m \in \omega\}$ is a convergent sequence in M converging to b_0. It implies that $S = f(C)$ with C being a convergent sequence in M, i.e, f is a sequence-covering map.

By Theorem 2.19 we get nice characterizations of s-images of metric spaces which were obtained in [9] as follows.

Corollary 2.20. {[9] Theorem 1.1} The following are equivalent for a space X.

(1) X is a sequence-covering s-image of a metric space,

(2) X has a point-countable cs-network.

Proof. (1) $\Rightarrow$ (2). Let $f : M \to X$ be a sequence-covering s-map from a metric space M onto X. Since M is metric, M has a σ-locally finite base $\mathcal{B}$. Then $f(\mathcal{B})$ is a point-countable cs-network for X.

(2) $\Rightarrow$ (1). Let $\mathcal{P}$ be a point-countable cs-network for X. It follows from Remark 1.7 that the Ponomarev-system $(f, M, X, \mathcal{P})$ exists.

By Lemma 2.9, $\mathcal{P}$ is a strong cs-network for X. From Theorem 2.19 and Lemma 2.15 we get that f is a sequence-covering s-map. It implies that X is a sequence-covering s-image of a metric space.

Corollary 2.21. {[9], Theorem 1.4} The following are equivalent for a space X.

(1) X is a sequence-covering, compact-covering quotient s-image of a metric space,

(2) X is a sequential space having a point-countable cs-network.

Proof. (1) $\Rightarrow$ (2). From Corollary 2.20, we only need to prove that X is sequential. It is clear because f is a quotient map from a metric space onto X.

(2) $\Rightarrow$ (1). Let $\mathcal{P}$ be a point-countable cs-network for X. As in the proof of Corollary 2.20, then X is an image of a metric space M under a sequence-covering s-map f where $(f, M, X, \mathcal{P})$ is a Ponomarev-system. It follows from Lemma 2.12, Lemma 2.13 and Lemma 2.14 that f is quotient. We shall prove that f is compact-covering by showing that $\mathcal{P}$ is a point-countable cfp-network for X. Let K be a compact subset of X. Since X is sequential, K is sequential compact. From Proposition 1.2 in [3], $\mathcal{P}_K = \{P \cap K : P \in \mathcal{P}\}$ is a point-countable k-network for K. It follows from Theorem 3.3 in [6] that K is metrizable. Then $\mathcal{P}$ is a point-countable cfp-network for X by Lemma 2.10. From Corollary 2.16, f is compact-covering.

The following is routine.

Corollary 2.22. The following are equivalent for a space X.

(1) X is a sequence-covering, compact-covering pseudo-open s-image of a metric space,

(2) X is a Fréchet space having a point-countable cs-network.

Moreover we get a mapping theorem on $\aleph$-spaces which belongs to [7].

Corollary 2.23. {[7], Theorem 3} The following are equivalent for a space X.

(1) X is an $\aleph$-space,

(2) X is a sequence-covering, compact-covering σ-image of a metric space,

(3) X is a compact-covering σ-image of a metric space,

(4) X is a sequence-covering σ-image of a metric space,

(5) X is a pseudo-sequence-covering σ-image of a metric space,

(6) X is a subsequence-covering σ-image of a metric space,

(7) X is a sequentially-quotient σ-image of a metric space.

Proof. (1) $\Rightarrow$ (2). Since X is an $\aleph$-space, X has a σ-locally finite cs-network $\mathcal{Q} = \bigcup\{\mathcal{Q}_n : n \in \mathbb{N}\}$ and a σ-locally finite k-network $\mathcal{R} = \bigcup\{\mathcal{R}_n : n \in \mathbb{N}\}$ where every $\mathcal{Q}_n$ and $\mathcal{R}_n$ is locally finite and every elements of $\mathcal{Q}$ and $\mathcal{R}$ are closed. For each $n \in \mathbb{N}$ put $\mathcal{P}_n = \mathcal{Q}_n \cup \mathcal{R}_n$, then $\mathcal{P}_n$ is locally finite. Therefore X has a σ-locally finite cs- and k-network $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$. It follows from Remark 1.7 that the Ponomarev-system $(f, M, X, \mathcal{P})$ exists. We shall prove that f is a sequence-covering and compact-covering σ-map. From Lemma 2.9, $\mathcal{P}$ is a strong cs-network for X. Hence f is a sequence-covering map by Theorem 2.19. Since $\mathcal{P}$ is a σ-locally finite closed k-network, $\mathcal{P}$ is a cfp-network. Hence f is a compact-covering map by Corollary 2.16. To complete the proof it suffices to show that f is a σ-map. For each $b = (\beta_n^b) \in M$ where $\{P_{\beta_n^b} : n \in \mathbb{N}\}$ is a network at some x_b in X, put $U_n^b = \{c = (\gamma_i) \in M : \gamma_i = \beta_i^b \text{ for all } i \leqslant n\}$, for each $n \in \mathbb{N}$ and $\mathcal{B}_b = \{U_n^b : n \in \mathbb{N}\}$. It follows from Lemma 2.18 that $\mathcal{B} = \bigcup\{\mathcal{B}_b : b \in M\}$

is a base for M and $f(U_n^b) = \bigcap_{i=1}^{n} P_{\beta_i^b}$. Since $\mathcal{P}$ is σ-locally finite, $f(\mathcal{B})$ is a σ-locally finite. It implies that f is a σ-map.

(2) $\Rightarrow$ (3), (2) $\Rightarrow$ (4). Obviously.

(3) $\Rightarrow$ (5), (4) $\Rightarrow$ (5), (5) $\Rightarrow$ (6). From Lemma 2.13.

(6) $\Rightarrow$ (7). From Lemma 2.14.

(7) $\Rightarrow$ (1). Let $f : M \to X$ be a sequentially-quotient σ-map from a metric space M onto X. Since f is a σ-map, M has a base $\mathcal{B}$ such that $f(\mathcal{B})$ is a σ-locally finite collection of subsets of X. On the other hand f is sequentially-quotient, $f(\mathcal{B})$ is a σ-locally finite cs^*-network for X. It implies that X is an $\aleph$-space.

Next we give a necessary and sufficient condition such that f is a pseudo-sequence-covering map.

Theorem 2.24. Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system, then the following are equivalent.

(1) f is a pseudo-sequence-covering map,

(2) $\mathcal{P}$ is a strong cs^*-network for X.

Proof. (1) $\Rightarrow$ (2). Let S be a convergent sequence in X, we shall prove that there is a countable subfamily $\mathcal{P}_S$ of $\mathcal{P}$ such that $\mathcal{P}_S$ is a cs^*-network for S in X. Indeed, since f is pseudo-sequence-covering, there exists a compact subset K in M such that $S = f(K)$. Put $\Lambda = \bigcup\{p_n(K) : n \in \mathbb{N}\}$, where every $p_n : \prod_{n\in\mathbb{N}} \Gamma_n \to \Gamma_n$ is a projection, then Λ is countable. Let

$$\mathcal{P}_S = \big\{ \bigcap_{\beta\in\Delta} P_\beta : \Delta \subset \Lambda, \Delta \text{ is finite}\big\}.$$

Since $\mathcal{P}$ is closed under finite intersections, $\mathcal{P}_S$ is a countable subfamily of $\mathcal{P}$. We shall prove that $\mathcal{P}_S$ is a cs^*-network for S in X. Let L be a convergent sequence in S converging to $x \in S \cap U$ with U open in X. As in the proof of Lemma 2.14, there exists a convergent subsequence $T \subset K$ in M such that $f(T)$ is a convergent subsequence of L. We get that T converges to $b \in f^{-1}(x) \subset f^{-1}(U)$. Let $b = (\beta_n)$. For each $n \in \mathbb{N}$ put

$$U_n = \{c = (\gamma_i) \in M : \gamma_i = \beta_i \text{ for all } i \leqslant n\}.$$

It follows from Lemma 2.18 that $\{U_n : n \in \mathbb{N}\}$ is a base at b in M. Since T converges to $b \in f^{-1}(U)$ which is open in M, T is eventually in some U_n with $U_n \subset f^{-1}(U)$. Therefore L is eventually in $f(U_n) \subset U$. From Lemma 2.18 $f(U_n) = \bigcap_{i=1}^{n} P_{\beta_i}$, and moreover $\bigcap_{i=1}^{n} P_{\beta_i}$ is an element of $\mathcal{P}_S$, we get that $\mathcal{P}_S$ is a cs^*-network for S in X.

(2) $\Rightarrow$ (1). Let $S = \{x_m : m \in \omega\}$ be a convergent sequence in X converging to x_0. We shall prove that $S = f(K)$ for some compact subset K in M. Assume that all x_m's are distinct. Since $\mathcal{P}$ is a strong cs^*-network for X, there exists a countable subfamily $\mathcal{P}_S$ of $\mathcal{P}$ such that $\mathcal{P}_S$ is a cs^*-network for S. It follows from Lemma 2.4 that there exists a subfamily $\mathcal{F}$ of $\mathcal{P}_S$ such that $\mathcal{F}$ has property $cs^*(S, X)$. Since $\mathcal{P}_S$ is countable, $\{\mathcal{F} \subset \mathcal{P}_S : \mathcal{F} \text{ has property } cs^*(S, X)\}$ is countable by finiteness of $\mathcal{F}$. Put

$$\{\mathcal{F} \subset \mathcal{P} : \mathcal{F} \text{ has property } cs^*(S, X)\} = \{\mathcal{F}_n : n \in \mathbb{N}\},$$

and put $\mathcal{F}_n = \{P_\beta : \beta \in \Delta_n\}$ for every $n \in \mathbb{N}$ where Δ_n is a finite subset of Γ_n.

Put
$$K = \big\{b = (\beta_n) \in \prod_{n\in\mathbb{N}} \Delta_n : \bigcap_{n\in\mathbb{N}} (P_{\beta_n} \cap S) \neq \emptyset\big\}.$$
Then we have that.

(a) $K \subset M$ and $f(K) \subset S$.

Let $b = (\beta_n) \in K$, then $\bigcap_{n\in\mathbb{N}} (P_{\beta_n} \cap S) \neq \emptyset$. Thus there is $x_m \in S$ such that $x_m \in \bigcap_{n\in\mathbb{N}} P_{\beta_n}$. We shall prove that $\{P_{\beta_n} : n \in \mathbb{N}\}$ is a network at x_m in X. Indeed, let U be an any open neighborhood of x_m in X. We consider two following cases.

(i) $x_m = x_0$.

We get that $U \cap S$ is a convergent sequence in X and $(U \cap S) \subset U$. It follows from Lemma 2.4 that there exists a subfamily $\mathcal{F}$ of $\mathcal{P}_S$ such that $\mathcal{F}$ has property $cs^*(U\cap S, U)$. Since $S - (U \cap S)$ is finite, $S - (U \cap S) = \{x_{n_i} : i = 1, \cdots, l\}$ for some $l \in \mathbb{N}$. For every $i = 1, \cdots, l$, note that $X - \big(S - (\{x_{n_i}\})\big)$ is an open neighborhood for x_{n_i} in X, there exists $P_i \in \mathcal{P}_S$ such that $x_{n_i} \in P_i \subset \big(X - (S - \{x_{n_i}\})\big)$. It is easy to see that $\mathcal{F} \cup \{P_i : i = 1, \ldots, l\}$ has property $cs^*(S, X)$. So there exists $k \in \mathbb{N}$ such that $\mathcal{F} \cup \{P_i : i = 1, \ldots, l\} = \mathcal{F}_k$. Since $x_0 \notin P_i$ for all $i = 1, \ldots, l$ and $x_0 \in P_{\beta_k} \in \mathcal{F}_k$, then $P_{\beta_k} \in \mathcal{F}$. Note that $\mathcal{F}$ has property $cs^*(S \cap U, U)$, $P_{\beta_k} \subset U$. It implies that $x_m \in P_{\beta_k} \subset U$.

(ii) $x_m \neq x_0$.

We get that $S - \{x_m\}$ is a convergent sequence in X and $S - \{x_m\} \subset X - \{x_m\}$ with $X - \{x_m\}$ open. It follows from Lemma 2.4 that there exists a subfamily $\mathcal{F}$ of $\mathcal{P}_S$ such that $\mathcal{F}$ has property $cs^*\big(S - \{x_m\}, X - \{x_m\}\big)$. Note that $\big(U - (S - \{x_m\})\big)$ is an open neighborhood of x_m, so there exists $P_m \in \mathcal{P}_S$ such that $x_m \in P_m \subset \big(U - (S - \{x_m\})\big) \subset U$. It is easy to see that $\mathcal{F} \cup \{P_m\}$ has property $cs^*(S, X)$. So there exists $k \in \mathbb{N}$ such that $\mathcal{F} \cup \{P_m\} = \mathcal{F}_k$. Since $x_m \notin F$ for every $F \in \mathcal{F}$ and $x \in P_{\beta_k} \in \mathcal{F}_k$, then $x_m \in P_{\beta_k} = P_m \subset U$.

From the above arguments (i) and (ii) we get that $\{P_{\beta_n} : n \in \mathbb{N}\}$ is a net work at x_m for every $m \in \omega$. It implies that $b \in M$ and $f(b) = x_m \in S$, i.e, $K \subset M$ and $f(K) \subset S$.

(b) $S \subset f(K)$.

Let $x \in S$. For every $n \in \mathbb{N}$, pick $\beta_n \in \Delta_n$ such that $x \in P_{\beta_n}$, then $x \in \bigcap_{n\in\mathbb{N}} (P_{\beta_n} \cap S) \neq \emptyset$. Put $b = (\beta_n)$, then $b \in K$. Using the argument in (a) we get $x = f(b)$. It implies that $S \subset f(K)$.

(c) K is a compact subset of M.

Since K is a subset of $\prod_{n\in\mathbb{N}} \Delta_n$ and $\prod_{n\in\mathbb{N}} \Delta_n$ is a compact subset by finiteness of each Δ_n, it suffices to prove that K is closed in $\prod_{n\in\mathbb{N}} \Delta_n$. Let $b = (\beta_n) \in \prod_{n\in\mathbb{N}} \Delta_n - K$, then $\bigcap_{n\in\mathbb{N}} (P_{\beta_n} \cap S) = \emptyset$. Note that $P_{\beta_n} \cap S$ is closed by property $cs^*(S, X)$ of $\mathcal{F}_n$ for every $n \in \mathbb{N}$. Since S is compact in X and $\bigcap_{n\in\mathbb{N}} (P_{\beta_n} \cap S) = \emptyset$, $\bigcap_{n=1}^{n_0} (P_{\beta_n} \cap S) = \emptyset$ for some $n_0 \in \mathbb{N}$. Put
$$W = \big\{c = (\gamma_n) \in \prod_{n\in\mathbb{N}} \Delta_n : \gamma_n = \beta_n \text{ if } n \leqslant n_0\big\},$$
then W is an open neighborhood of b in $\prod_{n\in\mathbb{N}} \Delta_n$ with $W \cap K = \emptyset$. So K is a closed subset of

$\prod_{n\in\mathbb{N}} \Delta_n$. It implies that K is compact.

From (a), (b) and (c) we get that $S = f(K)$ with K compact in M. It implies that f is a pseudo-sequence-covering map.

In the following part, we give a necessary and sufficient condition such that f is a subsequence-covering map or a sequentially-quotient map.

Theorem 2.25. Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system, then the following are equivalent.

(1) f is a subsequence-covering map,

(2) f is a sequentially-quotient map,

(3) $\mathcal{P}$ is a pseudo-strong cs-network for X,

(4) $\mathcal{P}$ is a pseudo-strong cs^*-network for X.

Proof. (1) $\Rightarrow$ (2). From Lemma 2.14.

(2) $\Rightarrow$ (3). Let S be a convergent sequence in X, we shall prove that there is a countable subfamily $\mathcal{P}_S$ of $\mathcal{P}$ such that $\mathcal{P}_S$ is a cs-network in X for some subsequence T of S. Indeed, since f is sequentially-quotient, there exists a convergent sequence C in M such that $T = f(C)$ is a convergent subsequence of S. Put $\Lambda = \bigcup\{p_n(C) : n \in \mathbb{N}\}$, where every $p_n : \prod_{n\in\mathbb{N}} \Gamma_n \to \Gamma_n$ is a projection, then Λ is countable. Let

$$\mathcal{P}_S = \big\{ \bigcap_{\beta\in\Delta} P_\beta : \Delta \subset \Lambda, \Delta \text{ is finite}\big\}.$$

Since $\mathcal{P}$ is closed under finite intersections, $\mathcal{P}_S$ is a countable subfamily of $\mathcal{P}$. As in the proof (1) $\Rightarrow$ (2) of Theorem 2.19 we get that $\mathcal{P}_S$ is a cs-network for T in X. It implies that $\mathcal{P}$ is a pseudo-strong cs-network for X.

(3) $\Rightarrow$(4). From Remark 2.6.

(4) $\Rightarrow$(1). Let S be a convergent sequence in X, we shall prove that there is a compact subset K in M such that $f(K)$ is a convergent subsequence of S. Indeed, since $\mathcal{P}$ is a pseudo-strong cs^*-network for X, there exists a countable subfamily $\mathcal{P}_S$ of $\mathcal{P}$ such that $\mathcal{P}_S$ is a cs^*-network in X for some convergent subsequence T of S. As in the proof (2) $\Rightarrow$ (1) of Theorem 2.24 we get that there is a compact subset K in M such that $f(K) = T$. It implies that f is a subsequence-covering map.

By Theorem 2.24 and Theorem 2.25 we get a nice characterization of pseudo-sequence-covering (subsequence-covering, sequentially-quotient) s-images of metric spaces as follows.

Corollary 2.26. The following are equivalent for a space X.

(1) X is a pseudo-sequence-covering s-image of a metric space,

(2) X is a subsequence-covering s-image of a metric space,

(3) X is a sequentially-quotient s-image of a metric space,

(4) X has a point-countable cs^*-network.

Proof. (1) $\Rightarrow$ (2). From Lemma 2.13.

(2) $\Rightarrow$ (3). From Lemma 2.14.

(3) $\Rightarrow$ (4). Let $f : M \to X$ be a sequentially-quotient s-map from a metric space M onto X. Since M is metric, M has a σ-locally finite base $\mathcal{B}$. Then $f(\mathcal{B})$ is a point-countable cs^*-network for X.

(4) $\Rightarrow$ (1). Let $\mathcal{P}$ be a point-countable cs^*-network for X. It follows from Remark 1.7 that the Ponomarev-system $(f, M, X, \mathcal{P})$ exists. By Lemma 2.11, $\mathcal{P}$ is a strong cs^*-network for X. From Theorem 2.24 and Lemma 2.15 we get that f is a pseudo-sequence-covering s-map. It implies that X is a pseudo-sequence-covering s-image of a metric space.

From Corollary 2.26 we get a nice characterization of pseudo-sequence-covering quotient s-images (quotient s-images) of metric spaces due to [6] and [13] as follows.

Corollary 2.27. {[6], Theorem 6.1 and [13], Theorem 2.3} The following are equivalent for a space X.

(1) X is a pseudo-sequence-covering quotient (resp., pseudo-open) s-image of a metric space,

(2) X is a quotient (resp., pseudo-open) s-image of a metric space,

(3) X is a sequential (resp., Fréchet) space having a point-countable cs^*-network.

Finally we consider a particular case when M is separable.

Lemma 2.28. Let $(f, M, X, \mathcal{P})$ be a Ponomarev-system, then the following are equivalent.

(1) M is separable,

(2) $\mathcal{P}$ is countable.

Proof. (1) $\Rightarrow$ (2). If $\mathcal{P}$ is not countable, then Γ is uncountable. For each $\beta \in \Gamma$ put $U_\beta = \{c = (\gamma_n) \in M : \gamma_1 = \beta\}$. Then each U_β is a non-empty open subset of M. We shall prove that $\{U_\beta : \beta \in \Gamma\}$ covers M. Indeed, if $c = (\gamma_n) \in M$, then $\{P_{\gamma_n} : n \in \mathbb{N}\} \subset \mathcal{P}$ is a network at $f(c)$. Pick $\beta = \gamma_1 \in \Gamma_1 = \Gamma$ then $c \in U_\beta$. It implies that $\{U_\beta : \beta \in \Gamma\}$ is an open cover for M. Since M is separable, $\{U_\beta : \beta \in \Gamma\}$ has a countable subcover. It is a contradiction because $U_\beta \cap U_\gamma = \emptyset$ whenever $\beta \neq \gamma$.

(2) $\Rightarrow$ (1). Obviously.

From the above we get a mapping theorem on $\aleph_0$-spaces which belongs to [4].

Corollary 2.29. {[4], Theorem 12} The following are equivalent for a space X.

(1) X is an $\aleph_0$-space,

(2) X is a sequence-covering, compact-covering image of a separable metric space,

(3) X is a compact-covering image of a separable metric space,

(4) X is a sequence-covering image of a separable metric space,

(5) X is a pseudo-sequence-covering image of a separable metric space,

(6) X is a subsequence-covering image of a separable metric space,

(7) X is a sequentially-quotient image of a separable metric space.

Proof. (1) $\Rightarrow$ (2). Since X is an $\aleph_0$-space, X has a countable cs-network $\mathcal{Q}$ and a countable k-network $\mathcal{R}$. Note that all elements of $\mathcal{Q}$ and $\mathcal{R}$ can be chosen closed. Put $\mathcal{P} = \mathcal{Q} \cup \mathcal{R}$, then $\mathcal{P}$ is a countable cs- and k-network for X. It follows from Remark 1.7 that the Ponomarev-system $(f, M, X, \mathcal{P})$ exists. Since $\mathcal{P}$ is countable, M is separable by Lemma 2.28. From Lemma 2.9, $\mathcal{P}$ is a strong cs-network for X. Hence f is a sequence-covering map by Theorem 2.19. On the other hand, $\mathcal{P}$ is a countable closed k-network, $\mathcal{P}$ is a cfp-network for X. Hence f is a compact-covering map by Corollary 2.16. It implies that f is a compact-covering and sequence-covering map from a separable metric space M onto X.

(2) $\Rightarrow$ (3), (2) $\Rightarrow$ (4). Obviously.

(3) $\Rightarrow$ (5), (4) $\Rightarrow$ (5), (5) $\Rightarrow$ (6). From Lemma 2.13.

(6) $\Rightarrow$ (7). From Lemma 2.14.

(7) $\Rightarrow$ (1). Let $f : M \to X$ be a sequentially-quotient map from a separable metric space M onto X. Since M is separable metric, M has a countable base $\mathcal{B}$. Then $f(\mathcal{B})$ is a countable cs^*-network for X. It implies that X is an $\aleph_0$-space.

Remark 2.30. If one of the above results contains "point-countable" and "s-image", then it is possible to replace them by "point-finite" and "compact image", "compact-countable" and "cs-image", or "locally countable" and "strong s-image" respectively. Therefore we get characterizations of compact images of metric spaces, characterizations of cs-images of metric spaces, and characterizations of strong s-images of metric spaces.

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Quasi–Pseudo–Metrization in $PqpM$–Spaces

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Abstract. A bitopologiacl space (X, T_P, T_Q) generated by probabilistic-quasi-pseudo-metric P is quasi-pseudo-metrizable if and only if there exists a quasi-pseudo-metric which induces those two topologies. In this paper, we consider a problem of quasi-pseudo-metrization of $PqpM$-spaces.

Keywords Probabilistic-quasi-pseudo-metric-space ($PqpM$-space), quasi-pseudo-metrization, bitopological space (X, T_P, T_Q), quasi-uniform structure.

§1. Introduction

A topological space (X, τ) is said to be metrizable if there exists a metric $d : X^2 \to \mathbb{R}$ which generates a topology T_d equivalent to τ.

For $PqpM$-spaces $(X, P, *)$, the problem leads to finding a quasi-pseudo-metric $p : X^2 \to \mathbb{R}$ such that it induces a topology T_p equivalent to T_P and its conjugate quasi-pseudo-metric q generates a topology T_q equivalent to T_Q. In other words, a bitopological space (X, T_P, T_Q) is quasi-pseudo-metrizable if and only if there exists a quasi-pseudo-metric p which induces those two topologies.

In section 2, we define the so-called (PE)-spaces and show their properties. Also, we show that a (PE)-space generates a quasi-uniform structure on the underlying set (cf., Fletcher and Lindgren [1]), which is quasi-pseudo-metrizable.

Next, we establish a number of relationships between (PE) and $PqpM$-spaces.

In section 3, some conditions for the quasi-pseudo-metrizability of a $PqpM$-space are given.

Finally, in section 4, we consider the problem of pseudo-metrization in $PqpM$-spaces.

§2. Preliminaries

We shall now consider functions defined on the extended real line $\overline{\mathbb{R}}$ with values in the complete lattice $(I, \leq)$. Note that the family $(I^{\overline{\mathbb{R}}}, \leq)$ is also a complete lattice. The following

subsets of $I^{\overline{\mathbb{R}}}$ will be used in the sequel

$$\Delta(\overline{\mathbb{R}}) = \{F \in I^{\overline{\mathbb{R}}}; F \text{ is nondecreasing, left-continuous on } \mathbb{R},$$
$$F(-\infty) = 0 \text{ and } F(+\infty) = 1\},$$
$$U(\overline{\mathbb{R}}) = \{u_a \in \Delta(\overline{\mathbb{R}});\ u_a = 1_{(a,+\infty)} \text{ for any } a \in \overline{\mathbb{R}}\}.$$

Note that $u_{-\infty}$ is the greatest element in $\Delta(\overline{\mathbb{R}})$ and $u_{+\infty}$ is the smallest in it. Furthermore, we define

$$\Delta^+(\overline{\mathbb{R}}) = \{F \in \Delta(\overline{\mathbb{R}}) : F(0) = 1\},$$
$$\Delta^+(\overline{\mathbb{R}}) = \{F \in \Delta^+(\overline{\mathbb{R}}) : \lim_{t\to+\infty} F(t) = 1\}.$$

The family $\Delta^+(\overline{\mathbb{R}})$ is a sublatices at $\Delta(\overline{\mathbb{R}})$ with the bounds $u_{\{-\infty\}}$ and u_0.

Definition 2.1. Let $(S, \leq, 0, 1)$ be a set partially ordered with bounds 0 and 1. A function $* : S^2 \to S$ is called a t_S-norm it the following condition holds: for all $a, b, c, d \in S$,

(S1) $a * b = b * a$,

(S2) $a * 1 = a$,

(S3) $(a * b) * c = a * (b * c)$,

(S4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Let $T(S, *)$ denote the family of all t_S-norm $*$ on the set S. Then the relation $\leq$ defined by the formula:

$$*_1 \leq *_2 \iff a *_1 b \leq a *_2 b, \quad \forall a, b \in S, \tag{2.1}$$

is a partially order in the family $T(S, *)$.

Second relation in the family $T(S, *)$ is defined as follows:

$$*_1 \gg *_2 \iff ((a *_2 c) *_1 (b *_2 d) \geq ((a *_1 b) *_2 (c *_1 d)), \quad \forall a, b, c, d \in S.$$

By putting $b = c = 1$ in (2.1), we obtain $a *_1 d \geq a *_2 d$ for $a, b \in S$ and hence $*_1 \geq *_2$ follows. Thus we know that $*_1 \gg x_2$ implies $*_1 \geq *_2$.

According to the Definition 2.1, a t_I-norm (see [2] and [3]) $T : I^2 \to I$ is in interval $I = [0, 1]$ an abelian semigroup with unit, and the t_I-norm T is nondecreasing with respect to each variable.

Definition 2.2. Let T be a t_I-norm.

(1) T is called a continuous t_I-norm if the function T is continuous with respect to the product topology on the set $I \times I$.

(2) T is said to be left-continuous if, for every $x, y \in (0, 1]$, the following holds:

$$T(x, y) = \sup\{T(u, v) : 0 < u < x, 0 < v < y\}.$$

(3) T is said to be-right-continuous if, for every $x, y \in [0, 1)$, the following conditions holds:

$$T(x, y) = \sup\{T(u, v) : x < u < 1, y, v < 1\}.$$

We shall now establish the notation related to a few most important t_I-norm defined in Definition 2.2:

(1) $M(x,y) = \min(x,y)$, for all $x, y \in I$. M is continuous and positive.

(2) $\Pi(x,y) = x \cdot y, x, y \in I$. π is strictly increasing and continuous.

(3) $W(x,y) = \max(x+y-1, 0)$ for all $x, y \in I$. W is continuous t_I-norm.

(4) $Z(x,y) = \begin{cases} x, & \text{if } x \in I \text{ and } y = I, \\ y, & \text{if } x = 1 \text{ and } y \in I, \\ 0, & \text{if } x, y \in (0,1). \end{cases}$

The function Z is right-continuous, but it fails to be left-condition. We give the following relations among t_I-norms defined above:

$$M \geq \Pi \geq W \geq Z, \tag{2.2}$$

$$M \gg \Pi \gg W \gg Z \tag{2.3}$$

We shall now present some properties of the t_S-norm defined on $\Delta^+(R)$. According to Definition 2.1. the ordered pair $(\Delta^+, *)$ is an abelian semigroup with the unit $u_0 \in \Delta^+$ and the operation $* : \Delta^+ \times \Delta^+ \to \Delta^+$ is a nondecreasing function. We notes that $u_\infty \in \Delta^+$ is a zero Δ^+. Indeed, by (S4) and (S2), we obtain

$$u_\infty \leq u_\infty * F \leq u_\infty * u_0 = u_\infty, \quad \forall F \in \Delta^+.$$

Let T be a left-continuous t_I-norm. Then the functiona $\Pi : \Delta^+ \times \Delta^+ \to \Delta^+$ defined by

$$T(F,G)(t) = T(F(t), G(t)), \quad \forall t \in [0, +\infty) \tag{2.4}$$

is a t_{Δ^+}-norm on the set Δ^+.

For every t_{Δ^+}-norm $*$, the following inequality holds $* \leq M$, where M is the t_I-norm of (1).

If T is left-continuous t_I-norm, then the function $*_T : \Delta^+ \times \Delta^+ \to \Delta^+$ defined by

$$F *_T G(t) = \sup\{T(F(u), G(s)) : u + s = t, u, s > 0\} \tag{2.5}$$

is a t_{Δ^+}-norm on Δ^+.

Let T be a continuous t_I-norm. Then the t_{Δ^+}-norms $*_T$ and T are uniformly continuous on (Δ^+, d_L), where d_L is a Modified Lévy metric (see [7, Definition 4.2.1]).

We finish this section by showing a few properties of the relation defined in Definition 2.2 in the context of t_{Δ^+}-norms. It T_1 and T_2 are continuous t_I-norms, then

$$T_1 \gg T_2 \iff *_{T_1} \gg *_{T_2}. \tag{2.6}$$

If T is a continuous t_I-norm and T is the t_{Δ^+}-norm of (2.5) then,

(i) $T \gg *_T$,

(ii) $M \gg *$ for all t_{Δ^+}-norms $*$.

§3. Probabilistic-quasi-pseudo-metric space ($PqpM$-space)

Definition 3.1. Let X be a nonempty set, $P : X^2 \to D$ and T in the t_I-norm. The triple (X, P, T) is called a quasi-pseudo-Menger space if it satisfies the following axioms:

(M1) $P_{xx} = u_0$ for all $x \in X$,

(M2) $P_{xz}(t_1 + t_2) \geq T(P_{xy}(t_1), P_{yz}(t_2))$ for all $x, y, z \in X$ and $t_1, t_2 > 0$.

If P satisfies also the additional condition:

(M3) $P_{xy} \neq u_0$ if $x \neq y$, the (X, P, T) is quasi-Menger space.

Moreover, if P satisfies the condition of symmetry $P_{xy} = P_{yx}$, then (X, P, T) is called a Menger-space.

Definition 3.2. Let (X, P, T) be a probabilistic quasi-Menger space (PqM) and the function $Q : X^2 \to D$ be defined by

$$Q_{xy} = P_{yx}, \quad \forall x, y \in X.$$

Then the ordered triple (X, Q, T) is also PqM-space. The function Q is called a conjugate Pqp-metric of the P. By (X, P, Q, T) we denote the structure generated by the Pqp-metric P on X.

Definition 3.3. ([2]) By probabilistic quasi pseudo-metric space ($PqpM$-space), we mean an ordered triple $(X, P, *)$, where X is a nonempty set, the operation $*$ is a t_{Δ^+}-norm and $P : X^2 \to D$ satisfies the following conditions: for all $x, y, z \in X$,

(P1) $P_{xx} = u_0$,

(P2) $P_{xz} \geq P_{xy} * P_y z$.

Generally, P is called a Pqp-metric. If P satisfies also the additional condition:

(P3) $P_{xy} \neq u_0$ if $x \neq y$, then $(X, P, *)$ is called a probabilistic quasi-metric space (PqM-space).

Note that if, moreover, P satisfies the condition of symmetry

(P4) $P_{xy} = P_{yz}$, for all $x, y \in X$, then $(X, P, *)$ is a probabilistic metric space (PM-space).

Definition 3.4. Let $(X, P, *)$ be a $PqpM$-space and the function $Q : X^2 \to D$ be defined by

$$Q_{xy} = P_{yx}, \quad \forall x, y \in X.$$

Then the triple (X, Q, x) is also a $PqpM$-space. The function Q is called a conjugate Pqp-metric of the P. Let $P(X, T, Q, *)$ denote the structure generated by the Pqp-metric P on X.

Definition 3.5. Let $(X, P, *)$ be $PqpM$-space. For all $x \in X$ and $t > 0$, a P-neighborhood of the point x is the set

$$U_x^P(t) = \{y \in X : P_{xy}(t) > 1 - t\} = \{y \in X : d_L(P_{xy,y_0}) < t\}. \tag{3.1}$$

Theorem 3.6. Let $(X, P, *)$ be $PqpM$-space ander a uniformly continuous t_{Δ^+}-norm $*$. Then the family $\{U_x^p(t) : t \in R^+\}_{x \in X}$ forms a complete system of neighborhoods in X (topolog τ_P on X).

Also, the Pqp-metric Q which is a conjugate of P generate a topology τ_Q. Thus the natural topological structure associated with a Pqp-metric is a bitopolocal space (X, τ_P, τ_Q).

Lemma 3.7. Let (X, P, Q, τ) be a structure defined by a Pqp-metric P. Suppose that $\tau \gg *$ and then function $F^2 : X^2 \to \Delta^+$ is defined by

$$F^{\tau}_{xy} = P_{xy} \tau Q_{xy} \quad \forall x, y \in X. \tag{3.2}$$

Then the ordered triple $(X, F^{\tau}, *)$ is a Probabilistic pseudo Metric-space. If additionally P satisfies the condition:

$$P_{xy} \neq u \quad \text{or} \quad Q_{zy} \neq 0, \quad \forall x, y \in X \ (x \neq y) \tag{3.3}$$

then $(X, F^{\tau}, *)$ is a probabilsitc metric space.

Remark 3.8. For an arbitrary Δ^+-norm $*$, we know that $M \gg \tau$ and we have

$$F^M(x, y) \geq F^{\tau}(x, y), \quad \forall x, y \in X. \tag{3.4}$$

The function F^M will be called the natural probabilistic pseudo-metric generated by the Pqp-metric P. It is the "greatest" among all the probabilistic pseudo-metrics generated by P.

§4. Properties of (PE)-spaces

Definition 4.1. (Fletcher and Lindgren, [1]) A quasi-uniform structure on a nonempty set X is a filter $\mathbb{U}$ in $X \times X$ satisfying the following conditions:

(a) Each element $U \in \mathbb{U}$ includes the set $\Delta = \{(x, x) : x \in X\}$,

(b) For every $U \in \mathbb{U}$, there exists $V \in \mathbb{U}$ such that $V \circ V \subset U$.

The pair $(X, \mathbb{U})$ is called a quasi-iniform space.

If $\mathbb{U}$ is a quasi-uniform structure in X, then the family $\mathbb{U}^{-1} = \{U^{-1} : U \in \mathbb{U}\}$ is also a quasi-uniform structure in X which is a conjugate of $\mathbb{U}$. A quasi-uniform structure $\mathbb{U}$ is a uniform structure if $\mathbb{U} = \mathbb{U}^{-1}$.

Definition 4.2. (Fletcher and Lindgren, [1]). A subfamily $\mathbb{B}$ of $\mathbb{U}$ is called a *base* of the quasi-uniform structure if, for all $U \in \mathbb{U}$, there exists a $V \in \mathbb{B}$ such that $V \subset U$.

A subfamily $\mathbb{S} \subset \mathbb{U}$ is called a subbase of $\mathbb{U}$ if the family of all finite intersections $U_1 \cap U_1 \cap \cdots \cap U_k$, where $U_i \in \mathbb{S}$ for $i = 1, 2, \cdots, k$, is a base of $\mathbb{U}$.

Example 4.3. The discrete quasi-uniformity on a set X in fact a uniformity: It is the quasi-uniformity $\mathcal{D}$ induced by the basis consisting of the diagonal set of $X \times X$ only. Generally, finite sets are implicitly equipped with the discrete uniformity.

A basis of a quasi-uniformity is symmetrical if all its elements are symmetrical. The quasi-uniformity generated by a symmetrical basis is actually a uniformity. If $\mathcal{B}$ is a basis of a quasi-uniformity $\mathcal{U}$, the entourages U^* for $U \in \mathcal{B}$ form a symmetrical basis of a uniformity $\mathcal{U}^*$ which is the smallest uniformity containing $\mathcal{U}$. In particular $\mathcal{U}^*$ is also generated by the entourages U^* for $U \in \mathcal{U}$.

Let $(X,\mathcal{U})$ be a quasi-uniform space. The intersection $\leq_{\mathcal{U}}$ of the elements of $\mathcal{U}$ forms a reflexive transitive relation on X that is a quasi order. We also denote by $\mathcal{U}$ the equivalence relation on X associated with the quasi order $\leq_{\mathcal{U}}$ and given by x $\Leftrightarrow_{\mathcal{U}}$ y if and only if x $\leq_{\mathcal{U}}$y and y $\leq_{\mathcal{U}}$x. If $\mathcal{U}$ is a uniformity then $\leq_{\mathcal{U}}$ and $\Leftrightarrow_{\mathcal{U}}$ coincide.

If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are quasi-uniform spaces, a mapping $f : X \to Y$ is said to be $(\mathcal{U}, \mathcal{V})-$ uniformly continuous (or uniformly continuous if there is no ambiguity) if, for each entourage $V \in \mathcal{V}$, there exists an entourage $U \in \mathcal{U}$ such that

$$(x, y) \in U \Rightarrow (f(x), f(y)) \in V.$$

In particular, such a mapping is monotonous if

$$x \leq_{\mathcal{U}} y \Rightarrow f(x) \leq_{\mathcal{V}} f(y).$$

For each $x \in X$, let

$$\mathcal{U}_x = \{U(x) : U \in \mathcal{U}\}.$$

There exists a unique topology on X, called the topology induced by $\mathcal{U}$, for which $\mathcal{U}_x$ is the filter of neighborhoods of x for each $x \in X$. Note that this topology is not necessarily Hausdorff: if the pair (x, y) of elements of X lies in each entourage U, that is, $x \leq_{\mathcal{U}} y$, then each neighborhood of x contains y.

We implicitly assume that the set $X \times X$ is endowed with the product topology.

Let $(X,\mathcal{U})$ be a uniform space. A filter $\mathcal{F}$ on X is a *Cauchy filter* if for each entourage $U \in \mathcal{U}$ there exists $F \in \mathcal{F}$ such that $F \times F \in V$ The uniform space $\left(X,\mathcal{U}\right)$ is said to be *complete* if each Cauchy filter on X is convergent. The Hausdorff completion $\left(\widehat{X},\widehat{\mathcal{U}}\right)$ of a uniform space $(X,\mathcal{U})$ and the uniformly continuous mapping on X are uniquely defined up to isomorphism by the following universal property: every uniformly continuous mapping f from $(X,\mathcal{U})$ into a Hausdorff complete uniform space $(Y,\mathcal{U})$ induces a unique uniformly continuous mapping $\widehat{f}$: $\widehat{X} \to Y$ such that $\widehat{f}(x) = f(x)$ for all $x \in X$.

Let us now consider a quasi-uniform space $(X,\mathcal{U})$. We define the Hausdorff completion $\left(\widehat{X},\widehat{\mathcal{U}}\right)$ of $(X,\mathcal{U})$ to be the Hausdorff completion of the uniform space $(X,\mathcal{U}^*)$.

Definition 4.4. A pair (X, E) is called a probabilistic E-space (shortly, (PE)-*space*) if X is a nonempty set and $E : X^2 \to \Delta^+$ is a function which satisfies the following conditions: for all $x, y, z \in X$,

$$E(x, x) = u_0, \tag{4.1}$$

For each $t > 0$, there exists $k > 0$ such that, if $E_{xy}(k) > 1 - k$ and $E_{xz}(k) > 1 - k$, then $E_{xz}(t) > 1 - t$. (4.2)

Theorem 4.5. Let (X, E) be a (PE)-space. Then the family $\mathbb{B} = \{U(t) : t > 0\}$, where $U(t)$ is defined by

$$U(t) = \{(x, y) \in X^2 : E_{xy}(t) > 1 - t\}, \tag{4.3}$$

is a base of a quasi-uniform structure on X.

Proof. For every $t > 0$, by (4.3), the diagonal Δ is a subset of $U(t)$. Next, let $0 < t_1 < t_2$ and let $(x, y) \in U(t_1)$. By (4.2) and the fact that $E_{xy} \in \Delta^+$ is nondecreasing, we infer that

$$E_{xy}(t_2) \geq E_{xy}(t_1) > 1 - t_2.$$

Thus we have $(x, y) \in U(t_2)$, which implies that $U(t_1) \subset U(t_2)$.

Finally, for all $t_1, t_2 > 0$, we obtain

$$U(\min(t_1, t_2)) = U(t_1) \cap U(t_2).$$

It follows directly from (4.2) that, for each $t > 0$, there exists $k > 0$ such that

$$U(k) \circ U(k) \subset U(t).$$

Thus the condition (b) of Definition 4.1 holds true. This completes the proof.

Corollary 4.6. If $\mathbb{U}$ is a quasi-uniform structure with a base $\mathbb{B}$ defined in Theorem 4.1, then the family

$$\mathbb{B}_\infty = \Big\{U\Big(\frac{1}{n}\Big) : n \in \mathbb{N}\Big\},$$

where $U\big(\frac{1}{n}\big)$ is defined by (4.3), is a countable base of $\mathbb{U}$.

Proof. It suffices to observe that, for every $t > 0$, there exists $n \in \mathbb{N}$ such that

$$U\Big(\frac{1}{n}\Big) \subset U(t).$$

Indeed, for every $t > 0$, there exists a natural number n such that

$$t \geq \frac{1}{n}.$$

Since the functions $E_{xy} \in \Delta^+$ are nondecreasing, we have

$$E_{xy}(t) \geq E_{xy}\Big(\frac{1}{n}\Big) > 1 - \frac{1}{n} > 1 - t.$$

This means that, if $(x, y) \in U\big(\frac{1}{n}\big)$, then it also belongs to $U(t)$, which shows that

$$U\Big(\frac{1}{n}\Big) \subset U(t), \quad \forall t > 0.$$

This completes the proof.

Lemma 4.7. Let (X, E) be a (PE)-space and let $E^{-1} : X^2 \to \Delta^+$ be defined by

$$E^{-1}(x, y) = E(y, x) \forall x, y \in X.$$

Then (X, E^{-1}) is also a (PE)-space that generates a quasi-uniform structure $\mathbb{U}^{-1}$ which is a conjugate of $\mathbb{U}$. A base of this structure is a countable family $\mathbb{B}_\infty^{-1}$ consisting of the sets of the form $U^{-1}(\frac{1}{n})$, where $U(\frac{1}{n}) \in \mathbb{B}_\infty$.

Lemma 4.8. Let (X, E) be a (PE)-space and let $E \vee E^{-1} : X^2 \to \Delta^+$ be given by

$$E \vee E^{-1}(x, y) = \min(E(x, y), E^{-1}(x, y)), \quad \forall x, y \in X.$$

Then the pair $(X, E \vee E^{-1})$ is a (PE)-space satisfying the symmetry condition, i.e., for all $x, y \in X$, $E \vee E^{-1}(x, y) = E \vee E^{-1}(y, x)$. Thus $E \vee E^{-1}$ generates a uniform structure on X for which the family $\mathbb{B} \cup \mathbb{B}^{-1}$ is a countable base.

Remark 4.9. Note that, in the proofs of Theorem 4.5 and Lemma 4.8, we have used the monotonicity of E_{xy}. However, we have not utilized the left-continuity of these functions. We thus conclude that the codomain of E of Definition 4.4 can be extended to the family Δ_1^+ of all nondecreasing functions $F : [0, +\infty] \to [0, 1]$ such that $F(0) = 0$ and $F(\infty) = 1$. The obtained function $E : X^2 \to \Delta_1^+$ also generates some quasi-uniform structure on X.

Theorem 4.10. Let $\mathbb{U}$ be a quasi-uniform structure defined by a (PE)-space (X, E). For all $x \in X$ and $U \in \mathbb{U}$, let $U[x] = \{y \in X : (x, y) \in U\}$. Then the family

$$T_E = \{\ G \subset X : \text{for each } x \in G, \text{ there exists } U \in \mathbb{U} \text{ such that } U[x] \subset G\}$$

forms a topology on X. The family

$$\mathbb{B}_T = \{\ G \subset X : \text{for each } x \in G, \text{ there is } U(t) \in \mathbb{B} \text{ such that } U(t)[x] \subset G\}$$

is a base of the topology T_E.

Proof. By the definition of T_E, it follows that a union of an arbitrary subfamily of T_E belongs to T_E. If $G_1, G_2 \in T_E$ and $x \in G_1 \cap G_2$, then there are $U_1, U_2 \in \mathbb{U}$ such that

$$U_1[x] \subset G_1, \quad U_2[x] \subset G_2.$$

By Definition 4.1, it follows that

$$U = U_1 \cap U_2 \in \mathbb{U}.$$

Since $U[x] = U_1 \cap U_2[x] \cap G_1 \cap G_2$, we have

$$G_1 \cap G_2 \in T_E,$$

which implies that T_E is a topology on X.

The second part of the proof immediately follows from the definition of a base of a topological space and from Definition 4.2. This completes the proof.

Theorem 4.11. Every quasi-uniform structure $\mathbb{U}$ generated from a space (X, E) is quasi-pseudo-metrizable. This means that there exists a quasi-pseudo-metric p which generates a quasi-uniform structure $\mathbb{U}$.

Proof. By Corollary 4.6, $\mathbb{U}$ has a countable base $\mathbb{B}_\infty$. The assertion now follows directly from a theorem by ([3], Theorem 13).

Remark 4.12. If p is a quasi-pseudo-metric which generates a quasi-uniform structure $\mathbb{U}$, then the conjugate structure $\mathbb{U}^{-1}$ generates a quasi-pseudo-metric q which is a conjugate of p. The uniform structure $\mathbb{U} \vee \mathbb{U}^{-1}$ of Lemma 4.8 generates a pseudo-metric $p \vee q = \max(p, q)$. It follows that, if $\mathbb{U} = \mathbb{U}^{-1}$, then the generating function is a pseudo-metric. In (PE)-spaces, this holds if and only if E satisfies the symmetry condition.

§5. Relationships between (PE) and $PqpM$-spaces

Relationships between probabilistic quasi-pseudo-metric spaces and probabilistic E-spaces are given by the following theorems.

Lemma 5.1. Let (X, P, T) be a quasi-pseudo-Menger space. Let the t_I-norm T satisfy the condition:

$$\sup\{T(x,x) : x < 1\} = 1. \tag{5.1}$$

Then (X, P) is a (PE)-space.

Proof. Let $t > 0$. Then, by (5.1), there is $k < \frac{t}{2}$ such that

$$T(1-k, 1-k) > 1-t.$$

By (P2), we get

$$\begin{aligned} P_{xz}(t) &\geq T\Big(P_{xy}\Big(\frac{t}{2}\Big), P_{yz}\Big(\frac{t}{2}\Big)\Big) \\ &\geq T(P_{xy}(k), P_{yz}(k)) \\ &\geq T(1-k, 1-k) \\ &> 1-t. \end{aligned}$$

This means that (2.3.2) holds true. Hence (X, P) is a (PE)-space. This completes the proof.

Lemma 5.2. Let $(X, P, *)$ be a $PqpM$-space. Assume additionally that the t_{Δ_+}-norm $*$ is continuous at the point (u_0, u_0). Then (X, P) is a (PE)-space.

Proof. It suffices to verify the triangle condition. Let $t > 0$. Then there exists $k > 0$ such that

$$d_L(G_1 * G_2, u_0) < t$$

whenever

$$d_L(G_1, u_0) < k$$

and

$$d_L(G_2, u_0) < k.$$

Then, by (P2) and (2.5), we infer that

$$d_L(P_{xy} * P_{yz}, u_0) < t.$$

This completes the proof.

An immediate consequence of Theorem 4.11 and Lemma 5.1 as well as of the Remark 4.12, we have the following:

Theorem 5.3. Let $(X, P, *)$ be a $PqpM$-space with t_{Δ^+}-norm $*$ being continuous at $(u_0, u_0) \in \Delta^+ \times \Delta^+$. Then the bitopological space $(X, T_{\mathbb{U}}, T_{\mathbb{U}^{-1}})$ generated from a quasi-uniform structure $\mathbb{U}$ is quasi-pseudo-metrizable. The topological space $(X, T_{\mathbb{U} \vee \mathbb{U}^{-1}})$ generated from the uniform structure $\mathbb{U} \vee \mathbb{U}^{-1}$ is pseudo-metrizable. This means that, if a quasi-pseudo-metric p

induces the topology $T_{\mathbb{U}}$, then the quasi-pseudo-metric q, which is a conjugate of p, induces $T_{\mathbb{U}^{-1}}$.

Lemma 5.4. If (X, P, T) is a quasi-pseudo-Menger space, then the bitopological space $(X, T_{\mathbb{U}}, T_{\mathbb{U}^{-1}})$ is quasi-pseudo-metrizable whenever the t_I-norm T satisfies the condition (5.1).

Remark 5.5. Note that (3.1) and (4.3) yield the equality

$$N_x^P(t) = U(t)[x], \quad \forall t > 0.$$

We thus infer that the bitopological spaces (X, T_P, T_Q) and $(X, T_{\mathbb{U}}, T_{\mathbb{U}^{-1}})$ defined, respectively, in Theorem 5.3 and Lemma 5.4 are identical and so we can replace the continuity assumption about the t_{Δ^+}-norm $*$ by the continuity assumption at the point (u_0, u_0).

On the other hand, as demonstrated by Muštari and Šerstnev ([5, Counterexample 6) for probabilistic metric spaces, if the t_{Δ^+}-norm is not continuous at (u_0, u_0), then the family of sets defined in (3.1) does not determine a topology on X, i.e., the following condition fails:

For each $y \in N_x^P(t)$, there exists $N_y^P(t_1)$ such that

$$N_y^P(t_1) \subset N_x^P(t).$$

Definition 5.6. For a probabilistic quasi-pseudo-metric space (X, P) and $s, t > 0$, we define a subset of X^2 by

$$U^P(s, t) = \{(x, y) \in X^2 : P_{xy}(s) > 1 - t\}. \tag{5.2}$$

Theorem 5.7. Let (X, P) be a probabilistic quasi-pseudo-metric space. Then the "quasi"-t_I-norm $T_{P\vee Q} = M(T_P, T_Q)$, where $T_P = I^2 \to I$ is defined by

$$T_P(a, b) = \inf\{P_{xz}(t_1 + t_2) : \quad P_{xy}(t_1) \geq a, \ P_{yz}(t_2) \geq b\}$$

and for all t_I-norms $T \leq T_{P\vee Q}$ has the property:

$$\sup\{T_{P\vee Q}(t, t) : 0 \leq t < 1\} = 1$$

if and only if, for each $t > 0$, there is $t_1 > 0$ such that, for all $s_1, s_2 > 0$,

$$U^P(s_1, t_1) \circ U^P(s_2, t_1) \subset U^P(s_1 + s_2, t). \tag{5.3}$$

Proof. For arbitrary $t > 0$, we select $t_1 > 0$ such that

$$T_{P\vee Q}(1 - t_1, 1 - t_1) > 1 - t.$$

Next, suppose that $P_{xy}(s) > 1 - t_1$ and $P_{yz}(s_2) > 1 - t_1$. Then, for $T_{P\vee Q}$, we obtain

$$\begin{aligned} P_{xz}(s_1 + s_2) &\geq T_{P\vee Q}(P_{xy}(s_1), P_{yz}(s_2)) \\ &\geq T_{P\vee Q}(1 - t_1, 1 - t_1) \\ &> 1 - t. \end{aligned}$$

On the other hand, for any $t > 0$, by (5.3), choose $t_1 > 0$. Then, for each $t_1 > k > 0$, let $P_{xy}(s_2) > 1 - k$ and $P_{yz}(s_2) > 1 - k$. Then we have

$$P_{xz}(s_1 + s_2) > 1 - t.$$

Hence, by Lemma 5.1, we have

$$T_{P \vee Q}(1 - k, 1 - k) > 1 - t.$$

This completes the proof.

Theorem 5.8. If (X, P) is a statistical quasi-pseudo-metric space, then the family $U = \{U^P(s, t) : s, t > 0\}$ is a base of a quasi-uniform structure on X if and only if, for each pair (s, t), there exists a pair (s_1, t_1) such that

$$U^P(s_1, t_1) \circ U^P(s_1, t_1) \subset U^P(s, t). \tag{5.4}$$

The fact that the quasi-uniform structure generated by U has a countable base yields the following:

Corollary 5.9. A quasi-uniform space (X, U) generated by a statistical quasi-pseudo-metrizable if and only if the condition (5.4) holds.

Remark 5.10. Let (X, P) be a statistical quasi-pseudo-metric space. Then the quasi-uniform structures generated by bases defined in (4.3) and (5.2) are equivalent.

Indeed, we have

$$U^P(t, t) = U(t), \quad U(\min(s, t)) \subset U^P(s, t), \quad \forall s, t > 0.$$

Remark 5.11. The following example shows that the condition (5.4) is essentially weaker than the condition (5.2). Let $X = [0, 1]$ and $P : X^2 \to \Delta^+$ be given by

$$P_{xy} = \begin{cases} u_{\frac{1}{2}}, & \text{if } x = 0 \text{ and } x \neq y, \\ u_{|x-y|}, & \text{otherwise.} \end{cases}$$

Let $s = \dfrac{8}{10}$ and $t > 0$. Then, for $x = \dfrac{1}{10}$ and $z = 1$, we have

$$P_{xy}(0, 2) = u_{\frac{1}{10}}(0, 2) = 1$$

and

$$P_{yz}(0, 6) = u_{\frac{1}{2}}(0, 6) = 1.$$

However, we have

$$P_{xz}\Big(\frac{2}{10} + \frac{6}{10}\Big) = u_{\frac{9}{10}}(0, 8) = 0.$$

Thus (5.3) does not hold.

Now let $s > 0$ and $t > 0$. It suffices to select numbers $s_1 > s_2$ and $t_1 = t$ in order for (5.4) to hold. Thus (X, P) is a (PE)-space by Remark 5.11. Notice also that (X, P) fails to be a statistical quasi-pseudo-metric space.

§6. Pseudo-metrization in $PqpM$-spaces

The problem of pseudo-metrization of probabilistic quasi-pseudo-metric spaces is characterized by the following result, which is a consequence of Theorem 5.3.

Theorem 6.1. Let $(X, P, *)$ be a $PqpM$-space such that P satisfies the symmetry condition and the t_{Δ^+}-norm $*$ is continuous at (u_0, u_0). Then the topology T_p is pseudo-metrizable. The topology T_P is metrizable if the function P satisfies the condition (P3) of Definition 3.3.

Proof. If P is symmetric, then we have the quasi-uniform structure $\mathbb{U} = \mathbb{U}^{-1}$ and $\mathbb{U} = \mathbb{U} \vee \mathbb{U}^{-1}$. The assertion is an immediate consequence of Theorem 5.3. For the proof of the second part of the theorem, notice that it follows from Definition 3.1 that $x \neq y$ if and only if $P_{xy} \neq u_0$. This means that

$$\bigcap_{t>1} U(t) = \Delta.$$

Indeed, let $x \neq y$. Then we have

$$d_L(P_{xy}, u_0) = t_1 > 0,$$

which implies that $(x, y) \notin U(t_1)$ and hence (x, y) does not belong to $\bigcap_{t>1} U(t)$. It follows that $\mathbb{U}$ is a Hausdorff uniform structure. Thus the topology generated by it is a Hausdorff topology. This completes the proof.

Remark 6.2. A metrization theorem for Menger spaces was proved by Schweizer and Sklar [6, Theorem 2]. Such a theorem for probabilistic metric spaces with a continuous t_I-norm $*$ goes back to Schweizer and Sklar [7, Theorem 12.1.6, p. 194].

Theorem 6.3. Let $(X, P, *)$ be a $PqpM$-space such that the t_{Δ^+}-norm $*$ is continuous at (u_0, u_0). If the topology T_P is stronger than T_Q, then (X, T_P) is pseudo-metrizable.

Proof. Let $T_{P\vee Q}$ be the topology generated by the probabilistic pseudo-metric $F_{P\vee Q}$. Topology $T_{P\vee Q}$ is the smallest topology containing T_P and T_Q. Thus we have

$$T_P = T_{P\vee Q}.$$

The assertion now follows immediately by the Definition 5.6. This completes the proof.

Theorem 6.4. Let $(X, P, *)$ be a $PqpM$-space such that the t_{Δ^+}-norm $*$ is continuous at (u_0, u_0). If P satisfies the condition:

$$\text{If } P_{xA} = u_0, \text{ then } Q_{xA} = u_0 \text{ for any } x \in X \text{ and } A \subset X, \tag{6.1}$$

then (X, T_P) is pseudo-metrizable.

Proof. Let A be Q-closed and let $z \in \overline{A}^P$. Then we have $P_{xA} = u_0$. By (6.1), it follows that also $Q_{xA} = u_0$, which means that $x \in \overline{A}^Q = A$. We infer that each Q-closed set A is P-closed. Thus T_P is stronger than T_Q. The assertion follows from Theorem 5.8. This completes the proof.

Theorem 6.5. Let $(X, P, *)$ be a $PqpM$-space such that the t_{Δ^+}-norm $*$ is continuous at (u_0, u_0). If $g_x : X \to [0, 1]$ defined by $g_x(y) = d_L(P_{xy}, u_0)$ is Q-continuous, then the space (X, T_0) is pseudo-metrizable.

Proof. If g_x is Q-continuous, then $N_x^P(t) \in T_Q$. Then T_Q is stronger than T_P by applying Theorem 5.8.

Theorem 6.6. Let $(X, P, *)$ be a $PqpM$-space such that the t_{Δ^+}-norm $*$ continuous. Then (X, T_P) is metrizable provided that (X, T_Q) is compact.

Proof. Let G be a Q-open set and take $y \in G$. The bitopological space (X, T_P, T_Q) generated by the probabilistic quasi-metric P is a pairwise Hausdorff space (see [8]). This means that, for every $x \in X - G$, there exist a Q-open set U and a P-open set V such that

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

Observe that the family $\{U : x \in X - G\}$ is a Q-open cover of $x - G$. By our assumption, there exists a finite subcover $\{U_1, \cdots, U_n\}$. Let $V = \bigcap\{V_i : i = 1, \cdots, n\}$. Then we have $y \in V \subset G$. Thus every Q-open set is also P-open, i.e., T_P is stronger than T_Q. The assertion now follows from Definition 5.6. This completes the proof.

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On Smarandache least common multiple ratio

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Abstract Smarandache LCM function and LCM ratio are already defined in [1]. This paper gives some additional properties and obtains interesting results regarding the figurate numbers. In addition, the various sequaences thus obtained are also discussed with graphs and their interpretations.

Keywords Smarandache LCM Function, Smarandache LCM ratio.

§1. Introduction

Definition 1.1. Smarandache LCM Function is defined as $SL(n) = k$, where $SL : N \longrightarrow N$
(1) n divides the least common multiple of $1, 2, 3, \cdots, k$,
(2) k is minimum.

Definition 1.2. The Least Common Multilpe of $1, 2, 3, \cdots, k$ is denoted by $[1, 2, 3, \cdots, k]$, for example $SL(1) = 1$, $SL(3) = 3$, $SL(6) = 3$, $SL(10) = 5$, $SL(12) = 4$, $SL(14) = 7$, $SL(15) = 5, \cdots$.

Definition 1.3. Smarandache LCM ratio is defined as

$$SL(n, r) = \frac{[n, n-1, n-2, \cdots, n-r+1]}{[1, 2, 3, \cdots, r]}.$$

Example.

$$SL(n, 1) = n,$$

$$SL(n, 2) = \frac{n.(n-1)}{2}, \ n \geq 2,$$

$$SL(n,3) = \begin{cases} \dfrac{n(n-1)(n-2)}{6}, & \text{if n is odd, } n \geq 3 \\ \dfrac{n(n-1)(n-2)}{12}, & \text{if n is even, } n \geq 3 \end{cases}$$

Proof. Here we use two results:
1. Product of LCM and GCD of two numbers = Product of these two numbers,
2. $[1, 2, 3, \cdots, n] = [[1, 2, 3, \cdots, p], [p+1, p+2, p+3, \cdots, n]]$.

Now,

$$SL(n,3) = \frac{[n, n-1, n-2]}{[1,2,3]}, \tag{1}$$

Here, $[n, n-1, n-2] = \left[n,\ \frac{(n-1)(n-2)}{(n-1, n-2)}\right]$.
But $(n-1, n-2) =$ GCD of n-1 and n-2, which is always 1.
Hence, $[n, n-1, n-2] = [n, (n-1)(n-2)]$ and clearly $[1,2,3] = 6$.

At $n = 3 : (1) \Rightarrow SL(3,3) = \frac{[3,2,1]}{[1,2,3]} = \frac{3\times 2\times 1}{6} = \frac{6}{6} = 1.$

At $n = 6 : (1) \Rightarrow SL(6,3) = \frac{[6,5,4]}{[1,2,3,4]} = \frac{6\times 5\times 4}{12} = 10.$

Hence, $SL(n,3) = \begin{cases} \frac{n(n-1)(n-2)}{6}, & \text{if n is odd} \\ \frac{n(n-1)(n-2)}{12}, & \text{if n is even} \end{cases}$

is proved.

Similarly $SL(n,4) = \begin{cases} \frac{[n, n-1, n-2, n-3]}{[1,2,3,4]}, & \text{for } n \geq 4 \\ \frac{n.(n-1).(n-2).(n-3)}{24}, & \text{if 3 does not divides n} \\ \frac{n.(n-1).(n-2).(n-3)}{72}, & \text{if 3 divides n} \end{cases}$

Similarly, $SL(n,5) = \frac{n.(n-1).(n-2).(n-3).(n-4)}{360}$, with other conditions also.
Here, we have used only the general valuesof LCM ratios given in ([2] and [3]).
The other results can be obtained similarly.

§2. Sets of $SL(n,r)$ [2]

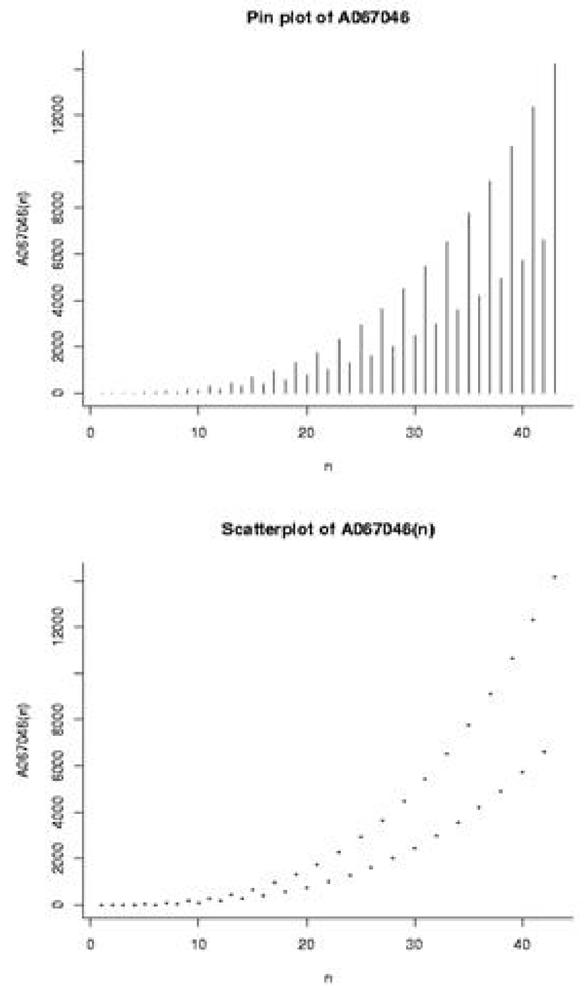
(1) $SL(n,1) = \{1,2,3,4,5,6,\cdots,n,\cdots\}$ It is a set of natural numbers.

(2) $SL(n,2) = \{1,3,6,10,\cdots,\frac{n(n-1)}{2},\cdots\}$ It is a set of triangular numbers.

(3) $SL(n,3) = \{1,2,10,10,35,28,84,\cdots,\frac{n(n-1)(n-2)}{12},\cdots\}$.

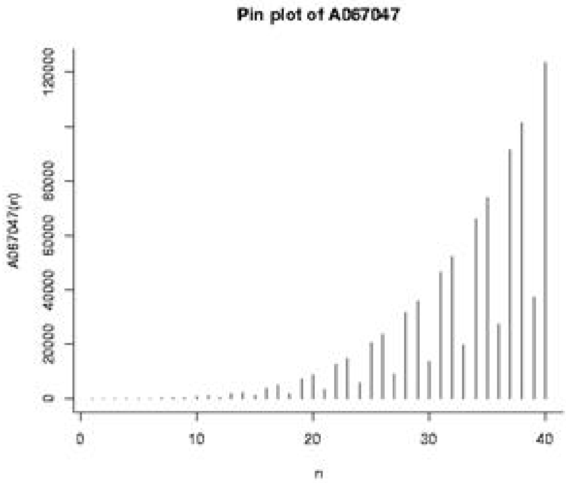
This set, with more elements, is $\{1,2,10,10,35,28,84,60,165,110,286,182,455,$
$280,680,408,969,570,1330,770,1771,1012,2300,1300,2925,1638,3654,2030,4495,2480,$
$5456,2992,6545,3570,7770,4218,9139,4940,10660,5740,12341,6622,14190,\cdots\}$.

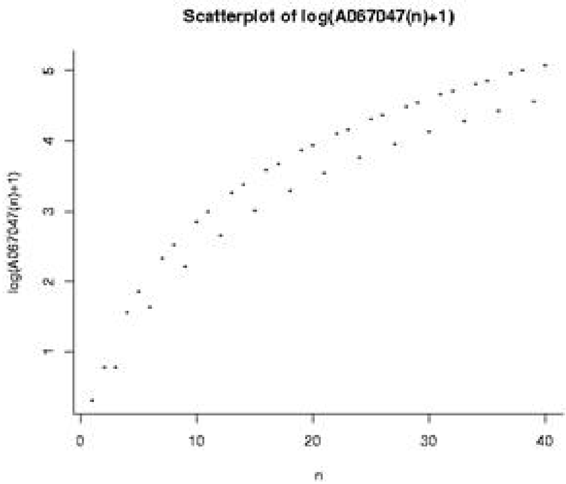
Its generating function is $\frac{x^4 + 2x^3 + 6x^2 + 2x + 1}{(1-x^2)^4}$.



Graph of SL(n,3)

Physical Interpretation of Graph of SL(n,3): This graph, given on the next page, represents the V-I characteristic of two diodes in forward bias mode. It is represented by the equation:
$I = I_0\{\exp(\frac{eV}{kBT}) - 1\}$, a rectifier equation, where,
I_0 = total saturation current,
e = charge on electron,
V = applied voltage,
kB = Boltzman's constant, and
T = temperature.
Here V is positive. X-axis represents voltage V and Y-axis is current in mA.
Also, this graph represents harmonic oscillator: Kinetic energy along Y-axis and velocity along X-axis.

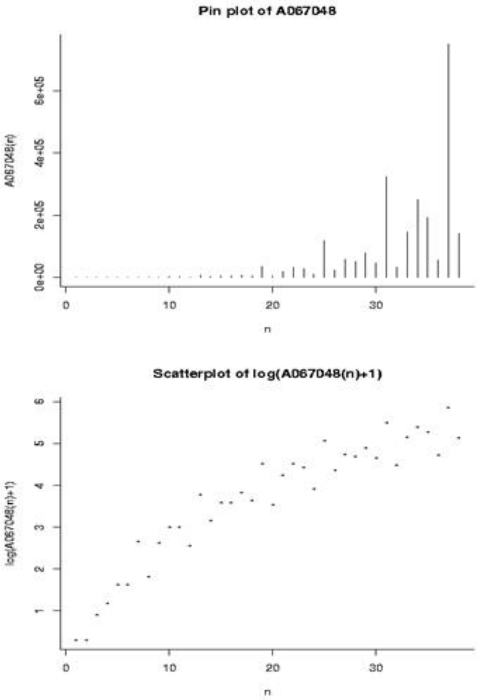




Graph of SL(n,4)

(4) $SL(n,4) = \{1, 5, 5, 35, 70, 42, \cdots, \dfrac{n(n-1)(n-2)(n-3)}{72}, \ldots\}$,
This set, to certain terms is $\{1, 5, 5, 35, 70, 42, 210, 330, 165, 715, 1001, 455,$
$1820, 2380, 1020, 3876, 4845, 1995, 7315, 8855, 3542, 12650, 14950, 5850, 20475, 23751, 9135,$
$31465, 35960, 13640, 46376, 52360, 19635, 66045, 73815, 27417, 91390, 101270, 37310, 123410, \cdots\}$.

Physical Interpretation of Graph of SL(n,4): This graph, the image of graph about a line of symmetry $y = x$, is a temperature-resistance characteristic of a thermister.
Its equation is $R = R_0.\exp[\beta(\frac{1}{T} - \frac{1}{T_0})]$, where
R_0 = resistance of room temperature,
R = resistance at different temperature,
β = constant,
T_0 = room temperature.



Graph of SL(n,5)

Temperature T, in Kelvin units, along X-axis and resistance R, in ohms, along Y-axis, β value lies between 3000 and 4000.

The above equation can be put as $R = C.e^{\frac{\beta}{T}}$.

Its another representation is potential energy (in ergs, along Y-axis) of system of spring against extension (in cetimeters along X-axis) for different weights.

The second graph below is characteristic curve of V_{CE} against I_{CE} at constant base current I_B.

(5) $SL(n, 5) = \{1, 1, 1, 7, 14, 42, 42, \cdots, \frac{n(n-1)(n-2)(n-3)(n-4)}{360}, \cdots\}$ This set, to certain terms, is $\{1, 1, 7, 14, 42, 42, 462, 66, 429, 1001, 1001, 364, 6188, 1428 \cdots\}$

Physical Interpretation of Graph of SL(n,5): The second graph of $\{SL(n, 5)\}$, given above, represents the V-I characteristic of two diodes in reverse bias mode. It is represented by the same equation mentioned in graph of $SL(n, 3)$ with a change that V is negative.

Hence, $-V \geq \frac{4kBT}{e}$, and that $\exp(\frac{-eV}{kBT}) \leq 1$, so that $I = I_0$.

This shows that the current is in reverse bias and remains constant at I_0, the saturation current, until the junction breaks down. Axes parameters are as above.

Similarly for the other sequences.

§3. Properties [3]

Murthy [1] formed an interserting triangle of the above sequences by writing them vertically, as follows:

1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	2	1						
1	5	10	10	5	1					
1	6	15	10	5	1	1				
1	7	21	35	35	7	7	1			
1	8	28	28	70	14	14	2	1		
1	9	36	84	42	42	42	6	3	1	
1	10	45	60	210	42	42	6	3	1	1

1. Here, the first column and the leading diogonal contains all unity.
The second column contains the elements of sequence $SL(n,1)$.
The third column contains the elements of sequence $SL(n,2)$.
The fourth column contains the elements of sequence $SL(n,3)$.
and similarly for other columns.
2. Consider that row which contains the elements 1 1 only as first row.
If p is prime, the sum of all elements of p^{th} row $\equiv 2(mod\ p)$.
If p is not prime, the sum of all elements of 4^{th} row $\equiv 2(mod\ 4)$.
The sum of all elements of 6^{th} row $\equiv 3(mod\ 6)$.
The sum of all elements of 8^{th} row $\equiv 6(mod\ 8)$.
The sum of all elements of 9^{th} row $\equiv 5(mod\ 5)$.
The sum of all elements of 10^{th} row $\equiv 1(mod\ 10)$.

§4. Difference

We have,
$SL(n,2) - SL(n-1,2) = SL(n-1,1)$.
This needs no verification.

$$\text{Also, } SL(n,3) - SL(n-1,3) = \frac{n(n-1)(n-2)}{6} - \frac{(n-1)(n-2)(n-3)}{6}$$
$$= \frac{(n-1)(n-2)}{2} = SL(n-1,2).$$

Similarly,

$$SL(n,4) - SL(n-1,3) = SL(n-1,3)$$

$$SL(n,5) - SL(n-1,5) = SL(n-1,4).$$

Hence, in general,

$$SL(n,r) - SL(n-1,r) = SL(n-1,r-1),\ r < n.$$

§5. Summation

Adding the above results, we get,

$$\sum_{r=2}^{\infty} SL(n,r) = n-1,\ n > 1.$$

§6. Ratio

We have,

$$\frac{SL(n,3)}{SL(n,2)} = \frac{n-2}{3},$$

$$\frac{SL(n,4)}{SL(n,3)} = \frac{n-3}{4}, \frac{SL(n,5)}{SL(n,4)} = \frac{n-4}{5}.$$

In general,

$$\frac{SL(n,r+1)}{SL(n,r)} = \frac{n-r}{r+1}.$$

§7. Sum of reciprocals of two cosecutive LCM ratios

We have,

$$\frac{1}{SL(n,2)} + \frac{1}{SL(n,3)} = \frac{n+1}{3 \cdot SL(n,3)},$$

$$\frac{1}{SL(n,3)} + \frac{1}{SL(n,4)} = \frac{n+1}{4 \cdot SL(n,4)},\ \frac{1}{SL(n,4)} + \frac{1}{SL(n,5)} = \frac{n+1}{5 \cdot SL(n,5)}.$$

In general,

$$\frac{1}{SL(n,r)} + \frac{1}{SL(n,r+1)} = \frac{n+1}{(r+1) \cdot SL(n,r+1)}.$$

§8. Product of two cosecutive LCM ratios

1. $SL(n,1) \cdot SL(n,2) = \dfrac{n^2(n-1)}{2!}$

2. $SL(n,2) \cdot SL(n,3) = \dfrac{n^2(n-1)^2(n-2)}{2! \cdot 3!}$

3. $SL(n,3) \cdot SL(n,4) = \dfrac{n^2(n-1)^2(n-2)^2(n-3)}{3! \cdot 4!}$

4. $SL(n,4) \cdot SL(n,5) = \dfrac{n^2(n-1)^2(n-2)^2(n-3)^2(n-4)}{4! \cdot 5!}$

In general,

$$SL(n,r) \cdot SL(n,r+1) = \frac{n^2 \cdot (n-1)^2 \cdot (n-2)^2 \cdot (n-3)^2 \cdot \cdots \cdot (n-r+1)^2 \cdot (n-r)}{r! \cdot (r+1)!}.$$

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Some New Relative Character Graphs

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Abstract In this paper based on the group characters there are certain graphs constructed. The properties are analysed with examples.

Keywords $IrrG$, RC-graphs, $Core_G H$

§1. Introduction

This paper is the outcome of the author in throwing some more light on the relative character Graphs $\Gamma(G, H)$ (RC-graphs) originally studied by T. Gnanaseelan in his Ph.D work [2].

It may be recalled that in the early 1940's R. Brauer while studying the 'ordinary' (complex) irreducible representation and the ρ-modular irreducible representation of a finite group G, constructed a finite simplex graph which was later called the Brauer Graph $B(G)$ and was extensively studied especially when G is a finite Chevalley group, or more generally, a simple group of his type [1].

Both these graphs have the same vertex set namely, the set $IrrG$ of complex irreducible Characters of G. In the Brauer group case, case vertices Φ, ψ are adjacent if and only if their reduction modulo a prime ρ dividing $O(G)$ contains at least one ρ-modular irreducible character in common. In the case of the RC-graph Φ and ψ are meant to be adjacent if and only if the restriction Φ_H and ψ_H to a given subgroup H of G contain at least one irreducible character of H in common. Clearly $\Gamma(G, H)$ is a simple graph.

We shall highlight some of the basic properties obtained by Gnanaseelan and others. This paper is a continuation of the earlier works deals with connectivity properties of the complement $\Gamma(G, H)$.

§2. Basic properties of RC-graphs

We need the following base minimum from character theory. For details, we refer to M. Isaac's book [3].

Given two characters Φ, ψ of G, let $(\Phi, \psi) = 1/0(G)\Sigma\Phi(s)\psi(s)$. Then, Q is irreducible if and only if $(\Phi, \Phi) = 1$ and if Φ and ψ are two distinct irreducible, then $(\Phi, \psi) = 0$.

In terms of this scalar product the adjacent condition can be given as follows:
$\Phi, \psi \in IrrG$ are adjacent if and only if $(\Phi,\psi)_H = \dfrac{1}{O(H)} \sum_{s\in H} \Phi(s)\overline{(\psi(s))} > 0$

Induced characters: Given any character Q of a subgroup H, the induced character Q^G of G is defined as
$Q^G(s) = \dfrac{1}{O(H)} \sum g \in GQ^0(gsg^{-1})$, where

$$Q^0(gsg^{-1}) = Q(gsg^{-1}) \text{ if } gsg^{-1} \in H \text{ and}$$
$$= 0, \text{ otherwise.}$$

Frobenius Reciprocity Formula:
If $Q \in IrrH$ and $\Phi \in IrrG$, then $(\Phi_H, Q) = (\Phi, Q^G)_G$.

Proposition 1. Two elements Φ, ψ of $IrrG$ are adjacent iff $\Phi \in \psi_\chi$ where $\chi^G = 1_H$.

For details and proofs of the following results, we refer to [2], [4] and [8].

Proposition 2. (G, H) is connected if and only if $core_G H = (1)$ where $core_G H$ is the largest normal subgroup of G^G contained in H. In particular if G is simple or if H is simple, then $\Gamma(G, H)$ is connected.

For the above proposition and for further results, we need the path lemma: Given any F in the connected component containing $1_G, \Phi$ in connected to 1_G by a path of length s if and only if $\Phi \subset \chi^s$, $s \geq 1$.

Proposition 3. $\Gamma(G, H)$ is a tree if and only if $G = NH$ is a Frobenius group with kernel N and complement H and N is unique minimal elementary abelian of order $p^m(\rho, aprime)$ and $0(H) = p^m - 1$. In this case, the tree is always a 'star'.

Proposition 4. $\Phi, \psi \in IrrG$ lie in the same connected components of $\Gamma(G, H)$ if and only if $\Phi \subset \psi\chi^s$ for some integer $s \geq 1$.

Proposition 5. If $\Gamma(G, H)$ is connected then it is always triangulated . (A tree is 'trivially' triangulated)

§3. Complements of RC-graphs

If H is a subgroup of G, it is very rare that the complements $\overline{\Gamma}(G, H)$ of $\Gamma(G, H)$ is of the form $\Gamma(G, H)$ for some subgroup K.

Problem 1. Characterize all groups G with the property that there exists a pair of subgroups H and K such that

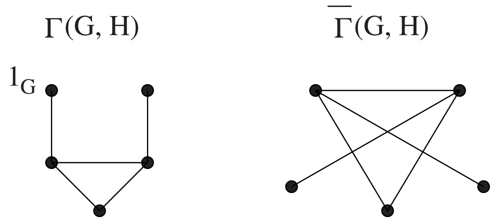
$$\Gamma(G, H) = \overline{\Gamma}(G, H).$$

In this connection the following results of Gnanaseelan is interesting proposition 6: If $g = NH$ is a semi direct product with N normal and H non-normal, then $\Gamma(G, N) = \overline{\Gamma}(G, H)$ if and only if G is Frobenius with kernel N and complement H.

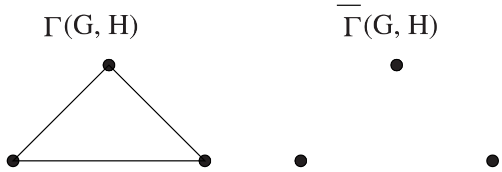
We shall now focus on the connectivity of $\Gamma(G, H)$ and $\overline{\Gamma}(G, H)$ (also see [9]).

At the outset, if a graph Γ is disconnected then Γ is connected. However, if Γ is connected, then Γ may be connected or not, as seen from the following examples:

$G = S_4, \ H = H_3$.



$G = S_3, \ H = (s)$.



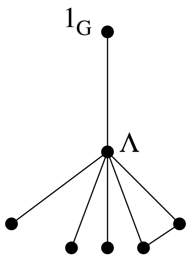
Problem 2. Given any finite group G, find all subgroups H such that $\overline{\Gamma}(G,H)$ is connected. (we can assume $core_G H = (1)$).

Problem 3. Find all finite groups G such that whenever $H \neq (1)$, then $\overline{\Gamma}(G,H)$ is connected.

Note that this includes the class of all finite abelian groups, because, if G is abelian of order g and H is a subgroup of order h, then $\Gamma(G,H)$ is disconnected with exactly h components each component being complete with g/h vertices.

Theorem 1. Let H be a subgroup of G such that the right action of G on G/H is doubly transitive. If $\Gamma(G,H)$ is not a tree and if $q <= n - 1C_2$, then $\overline{\Gamma}(G,H)$ is connected (where n and q are the number of vertices and number of edges of $\Gamma(G,H)$ respectively.)

Proof of Theorem 1. If $\Gamma(G,H)$ is disconnected then $\overline{\Gamma}(G,H)$ is always connected. Hence we assume that $\Gamma(G,H)$ is connected and is of the form:



Since the action of G on G/H is doubly Transitive,

$$\chi = 1_H^G = 1_G + \Lambda, \quad \Lambda \in IrrG.$$

Let J denote the sub graph $\Gamma(G,H) - 1_G$. Then J has $n-1$ vertices and $q-1$ edges. By Frobenuis reciprocity formula we have,

$$\Lambda_H = 1_H + \Sigma r_i Q_i, Q_i \in IrrH, r_i > 0$$

We shall prove that there is at least one $\Phi \in IrrG$, $\Phi \neq 1_G$ which is not adjacent to Λ.
For any $Q \in IrrH$, let $I(Q)$ denote the set of distinct irreducible characters occurring in Q^G.

First suppose that there exists $Q \in IrrH$ such that $I(Q) \subseteq J$ and σ_H is irreducible for every $\sigma \in I(Q)$. In other words, $\sigma_H = Q$ for every $\sigma \in I(Q)$. Then clearly any $\Phi \in I(Q)$ is such that Φ_H does not contain Q and hence $I(Q)$ is a connected component of $\Gamma(G, H)$, contradicting our assumption that $\Gamma(G, H)$, is connected.

Next, we shall prove that if $\sigma \in J$ is not Irreducible, then $\sigma \subset \sigma\Lambda$. In fact,

$$(\sigma, \sigma_x) = \frac{1}{O(G)} \sum_{s \in G} \sigma(s)\overline{\sigma(s)\chi(s)}$$

$$\frac{1}{O(G)} \left[\sum_{i=1}^{k} \sum_{s \in C_i} \sigma(s)\overline{\sigma(s)\chi(s)} \right]$$

(Where $C_1, C_2, \ldots, C_k$ are the conjugacy classes of G)

$$= \frac{1}{O(G)} \sum_{i=1}^{k} \sigma(s)\overline{\sigma(s)\chi(s)}|C_i|$$

(Where s runs through a set of class representatives)

$$= \frac{1}{O(G)} \sum_{i=1}^{k} \sigma(s)\overline{\sigma(s)}|C_i \cap H|\chi(1)$$

(using the relation $|C_i \cap H|\chi(1) = |Ci|\chi(s)$

$$\begin{aligned} &= \frac{1}{O(G)} \chi(1) \sum_{s \in H} \sigma(s)\overline{\sigma(s)} \\ &= \frac{\chi(1)}{O(G)} O(H)(\sigma, \sigma)_H \\ &= d > 1 \end{aligned}$$

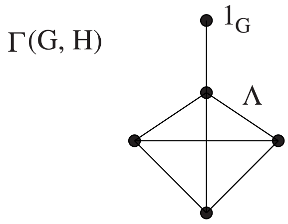
since σ_H is not irreducible using $\chi = 1_G + \Lambda$, this gives $(\sigma, \sigma\Lambda) > 0$, which means $\sigma \subset \sigma\Lambda$. Already by assumption σ and Λ are adjacent, which is by path lemma, gives $\sigma \subset \Lambda^2$. Using these 2 relations, we get $\sigma \subset \sigma\Lambda \subset \Lambda^3$. Hence by path lemma, we get a path of length 3 from σ to 1_G, which must pass through Λ. Hence we get a cycle $\Lambda\sigma\mu\Lambda$ for some $\mu \in J$. Repeating this process over and over again, we see that σ is adjacent to any Φ in J.

Now Suppose, for any $Q \in IrrH$, there exists $\sigma \in I(Q)$ such that σ_H is not irreducible. Then from the above argument, it follows that σ is adjacent to any Φ in J. Then J is a complete graph and hence $q = n - 1C_2 + 1$ (including the edge $1_G\Lambda$) which is a contradiction to our assumption that $q \leq n - 1C_2$.

Hence there must exist a vertex Φ not adjacent to Λ. In the complement, therefore, Φ and Λ are adjacent. Also, since Λ is the only vertex adjacent to 1_G in $\Gamma(G, H)$, all other vertices (including Φ) are adjacent to 1_G in $\overline{\Gamma}(G, H)$. Thus $\overline{\Gamma}(G, H)$ is connected.

Remarks 1.

1. The assumption that $q \leq n - 1C_2$ is necessary as we see from the following graph. Take $G = A_5$ and $H = A_4$.



Here $n = 5$ and $q = 7$, which is $> 4C_2 = 6$ of course $\overline{\Gamma}(G, H)$ is disconnected.

2. The condition that $\overline{\Gamma}(G, H)$ is not a tree is necessary, for otherwise, $\Gamma(G, H)$ would be a rooted tree (star) and hence Λ would be isolated in the complement.

Problem 4. Characterize all subgroups H of a given group G such that both $\Gamma(G, H)$ and $\overline{\Gamma}(G, H)$ are connected.

Theorem 2. $\overline{\Gamma}(G, H)$ is connected if and only if for every $\sigma \in I(1_H)$ there exists a $\Phi \in V - 1(1_H)$ such that σ and Φ are not adjacent in $\Gamma(G, H)$.

Proof of Theorem 2. First assume that $\overline{\Gamma}(G, H)$ is connected. Let $\sigma \in I(1_H)$. Suppose σ is adjacent to all $\Phi \in V - I(1_H)$ in $\Gamma(G, H)$. Since already σ is adjacent to all $\psi \in I(1_H)$, as $I(1_H)$ is complete, σ is adjacent to all vertices in $\Gamma(G, H)$. Hence σ is an isolated vertex in $\overline{\Gamma}(G, H)$, contradiction.

For the converse part, assume the given condition. Since 1_G is not adjacent to any vertex in $V - I(1_H)$, 1_G is adjacent to every vertex in $V - \overline{I}(1_H)$ in $\Gamma(G, H)$. By the assumption, if $\sigma \in I(1_H)$ and $\Phi \in V - I(1_H)$, σ and Φ are adjacent in $\Gamma(G, H)$ ....(1)

Also 1_G is adjacent to all vertices in $V - I(1_H)$ in $\overline{\Gamma}(G, H)$. In fact, no vertex Φ can be adjacent to 1_G without belonging to $I(1_H)$. Therefore, if $\Phi \in V - I(1_H)$, then Φ is not adjacent to 1_G in $\Gamma(G, H)$. Hence Φ is adjacent to 1_G in $\overline{\Gamma}(G, H)$ ....(2)

Now take any two α, β in γ.

Case 1: Both α and β belong to $I(1_H)$. By (2), there exist ξ_1, ξ_2 in $V - I(1_H)$, such that α is adjacent to ξ_1 and β is adjacent to ξ_2 in $\overline{\Gamma}(G, H)$. But by (1), 1_G is adjacent to both ξ_1 and ξ_2 in $\overline{\Gamma}(G, H)$. Then we get a path between α and β. (see figure 1).

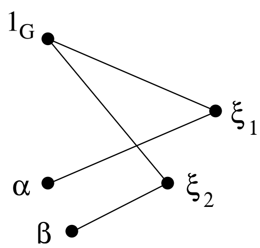


Figure 1:

Case 2: Let $\alpha \in I(1_H)$ and $\beta \in V - I(1_H)$, By (1), 1_G is adjacent to β in $\overline{\Gamma}(G, H)$, and by (2), their exists $\xi \in V - I(1_H)$ such that α is adjacent to ξ in $\overline{\Gamma}(G, H)$. This shows that there is a path between α and β in $\overline{\Gamma}(G, H)$. (see figure 2).

Case 3: Both α and β do not lie in $I(1_H)$. By (1), 1_G is adjacent to α and β in $\overline{\Gamma}(G, H)$; hence there is a path between α and β in $\overline{\Gamma}(G, H)$. (see figure 3).

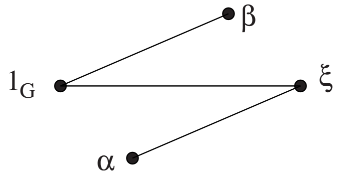


Figure 2:

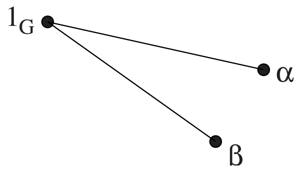


Figure 3:

The above arguments prove that $\overline{\Gamma}(G, H)$ is connected. Hence the theorem.

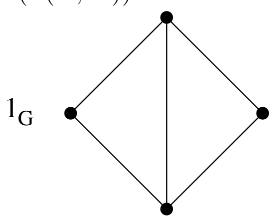
Theorem 3. Let the right action of G and G/H be doubly transitive. Then $\overline{\Gamma}(G, H)$ is connected if and only if $diam\Gamma(G, H) \geq 3$.

Proof of Theorem 3. Due to doubly transitivity, $I(1_H) = 1_G + \lambda$, $\Lambda \in IrrG$, and hence is the unique vertex adjacent to 1_G. First let $diam\Gamma(G, H) \geq 3$. Then their exists $\Phi \in V - I(1_H)$ whose distance from Λ is at least 2. Therefore, given $\Lambda \in I(1_H)$ we can find $\Phi \in V - I(1_H)$ such that Λ and Φ are not adjacent. Hence by theorem 2, $\overline{\Gamma}(G, H)$ is connected.

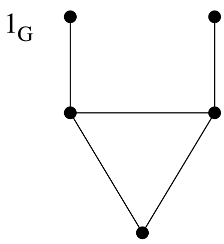
Conversely, assume that $\overline{\Gamma}(G, H)$ is connected. Hence by the same theorem, their exists $\Phi \in V - I(1_H)$ such that Λ and Φ are not adjacent. Therefore, $dist.(\Lambda, \Phi) \geq 2$. Hence $diam\Gamma(G, H) \geq 3$. This proves the theorem.

Proposition 6.

(i) If $G = NH$ is Frobenius, then $r(\Gamma(G, H)) = 1$. As an example $\Gamma(D_{10}, C_2)$ is the following.



Of course there are cases when $r(\Gamma(G, H)) > 1$. For instance, when $G = S_4$ and $H = S_3$, $\Gamma(G, H)$ is the graph.



Whose domination number is 2. For details on domination theory of RC-graphs, we refer

to [5], [8].

(ii) RC-graphs as signed graphs. There is a natural sign attached to the edges. If Φ, ψ are the vertices and $\Phi_H = \Sigma m_i G_i$ and $\psi_H = \Sigma n_i Q_i$, so that $(\Phi, \psi)_H = \Sigma m_i n_i$, the edge $\Phi\psi$ gets a + sign if $\Sigma m_i n_i$ is even and a - sign if $\Sigma m_i n_i$ is odd. The first author has initiated a study on RC-graphs as signed graphs following the work of B.D. Acharya et al. (see [6]).

(iii) Among the many products occurring in graph theory the one that suits our RC-graph is the 'strong direct product'. The domination number of a strong direct product cannot be greater than that of the more fancied Cartesian product

If Γ_1 and Γ_2 are two (connected) graphs then we have the famous vizings conjecture: $\gamma(\Gamma_1 \times \Gamma_2) \geq \gamma(\Gamma_1)\gamma(\Gamma_2)$. A variation of the above conjunctive for RC-graphs and strong direct products fails. (For details see [7]).

Conclusion

Graph - theorists have already shown interest in RC-graphs as they provide some relief from the monotonous 'edge - dot' study! The authors believe that the pictorial description of character theory could help people to understand representation theory in a better way. (The restriction behavior of an irreducible G - character to a subgroup H, apart from Clifford's theory when H is normal, is not completely understood). Last but not the least many new perhaps hitherto unseen, subgroups have already been located in an arbitrary finites groups using RC-graphs such as almost - normal subgroups, dominate subgroups, domination - free subgroups etc, whose definition and details are omitted here for want of time and space.

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Class A weighted composition operators

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Abstract In this paper Class A weighted composition operators on L^2 - spaces are characterized and their various properties are studied.

Keywords Composition operators, class A operators.

§1. Preliminaries

Let (X, Σ, λ) be a sigma-finite measure space and let $T : X \to X$ be a non-singular measurable transformation. Let $L^2 = L^2(X, \Sigma, \lambda)$. Then the composition transformation C_T is defined by $C_T f = f \circ \mathrm{T}$ for every f in $L^2(\lambda)$. If C_T happens to be a bounded operator on L^2, then we call it the composition operator induced by T.

C_T is a bounded linear operator on L^2 precisely when (i) the measure λT^{-1} is absolutely continuous with respect to λ and (ii) the Radon-Nikodym derivative $d\lambda T^{-1}/d\lambda$ is in $L^\infty(X, \Sigma, \lambda)$. Let $R(C_T)$ denote the range of C_T and C_T^*, the adjoint of C_T.

A weighted composition operator is a linear transformation acting on a set of complex valued Σ mesurable functions of the form $Wf = wf \circ T$, where w is a complex valued Σ measurable function. In case $w = 1$ a.e., W becomes a composition operator, denoted by C_T.

To examine the weighted composition operators efficiently, Lambert [1], associated with each transformation T, the so called conditional expectation operator $E(\bullet|T^{-1}\Sigma) = E(\bullet)$. More generally, $E(f)$ may be defined for bounded measurable function f or non-negative measurable functions f; for details on the properties of E, see [4], [5], [6].

As an operator on L^P, E is the projection onto the closure of the range of C_T. E is the identity on L^P if and only if $T^{-1}\Sigma = \Sigma$.

The Radon-Nikodym derivative of λT^{-1} with respect to λ is denoted by h and that of λT^{-k} with respect to λ is denoted by h_k where T^k is obtained by composing T with itself k times. Let w_k denote $w(w \circ T)$ $(w \circ \mathrm{T}^2) \cdots (w \circ \mathrm{T}^{k-1})$ so that $W^k \mathrm{f} = \mathrm{w}_k (\mathrm{f} \circ \mathrm{T})^k$.

§2. Class A composition operators

Let $B(H)$ denote the Banach algebra of all bounded linear operators on a Hilbert space H. An operator $T \in B(H)$ is said to be hyponormal if $T^*T \geq TT^*$, T is said to be p-hyponormal [2]

if $(T^*T)^p$ - $(TT^*)^p \geq 0$, $0 < p < 1$. For a p-hyponormal operator $T = U|T|$ Aluthge introduced the operator $\tilde{T}$= $|T|^{1/2}$ U $|T|^{1/2}$ which is called Aluthge transformation and Aluthge showed very interesting results on T. The operator T is said to be w - hyponormal [3] if $|\ \tilde{T}\ | \geq |T| \geq |\ \tilde{T}^*|$. T is paranormal if $\|T^2\mathrm{x}\ \|\ \|x\| \geq \|Tx\|^2$; T is quasihyponormal if $\|T^*T\| \leq \|T^2\mathrm{x}\|$ for all x in H or equivalently $T^{*2}T^2$ - $(T^*T)^2 \geq 0$.

An operator T belongs to class A [7] if $|T^2| \geq |T|^2$. Furuta, Masatoshi Ito and Takeaki Yamazaki have characterized class A operators as follows. An operator T belongs to class A if and only if $(T^*\ |T|^2T)^{1/2} \geq T^*T$.

§3. Previous results on M-paranormal composition operators

Panayappan and Veluchamy [9] have characterized M-paranormal composition operators as follows.

Theorem 3.1. Let $C_T \in \mathrm{B}(\mathrm{L}^2(\lambda)$. Then C_T is M-paranormal if and only if $M^2\mathrm{h}_0 + 2\mathrm{kf}_0 + \mathrm{k}^2 \geq 0$ a.e., for all $k \in R$, where f_0 is the Radon-Nikodym derivative of $\lambda\mathrm{T}^{-1}$ with respect to λ and h_0 is the Radon-Nikodym derivative of $\lambda(\mathrm{T}\circ\mathrm{T})^{-1}$ with respect to λ.

Also Panayappan [8] generalized the above result to the weighted composition operators as follows.

Theorem 3.2. W is M-paranormal if and only if $M^2\mathrm{h}_2\ \mathrm{E}(\mathrm{w}_2^2)\circ\mathrm{T}^{-2} \geq \mathrm{h}^2\ [E(w^2)\circ\mathrm{T}^{-1}]^2$ a.e..

The aim of the paper is to characterize class A weighted composition operators and also to show that class A and paranormal operators coincide in the case of weighted composition operators.

§4. Weighted class A composition operators

Theorem 4.1. W is of class A if and only if $h_2\mathrm{E}(\mathrm{w}_2^2)\circ\mathrm{T}^{-2} \geq h^2[\ \mathrm{E}(\mathrm{w}^2)\circ\mathrm{T}^{-1}]^2$ a.e..

Proof. We have, $W^k f = w_k(\mathrm{f}\circ\mathrm{T}^k)$ and $W^*\mathrm{f} = \mathrm{h}_k\ \mathrm{E}(\mathrm{w}_k\mathrm{f})\circ T^{-k}$

and so
$$\begin{aligned}\mathrm{W}^{*2}\mathrm{W}^2\mathrm{f} &= \mathrm{W}^{*2}(\mathrm{w}_2(\mathrm{f}\circ\mathrm{T}^2))\\ &= h_2\ \mathrm{E}(\mathrm{w}_2{}^2\mathrm{f}\circ\mathrm{T}^2)\circ\mathrm{T}^{-2}\\ &= h_2\ \mathrm{E}(\mathrm{w}_2^2)\circ\mathrm{T}^{-2}\mathrm{f}.\end{aligned}$$

Also $W^*Wf = hE(w^2)\circ T^{-1}f$.

Then W is of class A

$\Longleftrightarrow (\mathrm{W}^*\ |\mathrm{W}|^2\ \mathrm{W})^{1/2} \geq \mathrm{W}^*W$

that is $(W^*\ |\mathrm{W}|^2\ \mathrm{W}) \geq (\mathrm{W}^*\mathrm{W})^2$

if and only if $(\mathrm{W}^*(\mathrm{W}^*\mathrm{W})\mathrm{W}) \geq (\mathrm{W}^*\mathrm{W})^2$

$\Longleftrightarrow \mathrm{W}^{*2}\mathrm{W}^2 \geq (\mathrm{W}^*\mathrm{W})^2$

$\Longleftrightarrow \langle\ (\mathrm{W}^{*2}\mathrm{W}^2$ - $(\mathrm{W}^*\mathrm{W})^2)\mathrm{f},\ \mathrm{f})\ \rangle \geq 0$ for every $\mathrm{f} \in \Sigma$

$\Longleftrightarrow \int_E \mathrm{h}_2\ \mathrm{E}(\mathrm{w}_2^2)\circ\mathrm{T}^{-2}$ - $(\mathrm{h}\ \mathrm{E}(\mathrm{w}^2)\circ\mathrm{T}^{-1})^2\ |\mathrm{f}|^2\ d\lambda \geq 0$ for every $\mathrm{E} \in \Sigma$

$\Longleftrightarrow \mathrm{h}_2\mathrm{E}(\mathrm{w}_2^2)\circ\mathrm{T}^{-2} \geq \mathrm{h}^2(\mathrm{E}(\mathrm{w}^2)\circ\mathrm{T}^{-1})^2$ a.e.

Theorem 4.2. Let W be a weighted composition operator with weight $w > 0$. Then the following are equivalent

(i) W is paranormal;

(ii) W is class A;

(iii) W is quasihyponormal.

Proof. (i) $\Rightarrow$ (ii)

Suppose W is paranormal. Then by Theorem 2.3 [8]

h_2 $\mathrm{E}(\mathrm{w}_2^2)\circ\mathrm{T}^{-2} \geq \mathrm{h}^2$ $(\mathrm{E}(\mathrm{w}^2)\circ$ $\mathrm{T}^{-1})^2$ a.e.

W is class A.

(ii) $\Rightarrow$(iii)

Suppose W is class A. Then by Theorem 4.1, h_2 $E({w_2}^2)\circ\mathrm{T}^{-2} \geq h^2$ $(E(w^2)\circ\mathrm{T}^{-1})^2$ a.e..

Therefore by Corollary 2.2 [8], W is quasihyponormal.

(iii) $\Rightarrow$ (i)

Suppose W is quasihyponormal, then

h_2 $E(w_2^2) \circ T^{-2} \geq \mathrm{h}^2$ $(\mathrm{E}(\mathrm{w}^2) \circ \mathrm{T}^{-1})^2$ a.e.

Therefore by Corollary 2.4, [8], W is paranormal.

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Position vectors of some special space-like curves according to Bishop frame in Minkowski space E_1^3

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Abstract In this work, we investigate position vectors of some special space-like curves with respect to Bishop frame in E_1^3. A system of differential equation whose solution gives the components of position vector on the Bishop axis is established. Via its special solutions, some characterizations are presented.

Keywords Minkowski space, Bishop frame, space-like curve.

§1. Introduction and preliminaries

In the existing literature, it can be seen that, most of classical differential geometry topics have been extended to Lorentzian manifolds. In this process, generally, researchers used a standard moving Frenet frame. Some of kinematical models were adapted on this moving frame, due to tranformation matrix among derivative vectors and frame vectors. Thereafter, researchers aimed to have an alternative frame for curves and other applications. Bishop frame, which is also called alternative or parallel frame of the curves, was introduced by L.R. Bishop in 1975. And, this frame have been used in many research papers, in classical manner or Lorentzian manifolds, etc.

In this work, we consider a space-like curve with a space-like binormal. Then, with the notion of Bishop frame, we investigate position vectors of some special space-like curves in Minkowski space E_1^3.

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^3 are briefly presented. (A more complete elementary treatment can be found in [2] and [4].)

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since $\langle , \rangle$ is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian characters: it can be space-like if $\langle v, v\rangle > 0$ or $v = 0$, time-like if $\langle v, v\rangle < 0$ and null if $\langle v, v\rangle = 0$ and $v \neq 0$. Similarly, an

arbitrary curve $\varphi = \varphi(s)$ in E_1^3 can locally be space-like, time-like or null (light-like), if all of its velocity vectors φ' are respectively space-like, time-like or null (light-like), for every $s \in I \subset R$. The pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by $\|a\| = \sqrt{|\langle a, a\rangle|}$. φ is called an unit speed curve if velocity vector v of φ satisfies $\|v\| = \pm 1$. For vectors $v, w \in E_1^3$ it is said to be orthogonal if and only if $\langle v, w\rangle = 0$.

Denote by $\{T, N, B\}$ the moving Frenet frame along the curve φ in the space E_1^3. For a space-like curve φ with first and second curvature, κ and τ in the space E_1^3, the following Frenet formulae are given in [4].

Let φ be a space-like curve with a space-like binormal, then the Frenet formulae read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{1}$$

where

$$\langle N, N\rangle = -1,\ \langle T, T\rangle = \langle B, B\rangle = 1,$$
$$\langle T, N\rangle = \langle T, B\rangle = \langle T, N\rangle = \langle N, B\rangle = 0.$$

The Bishop frame is due to L.R. Bishop [3]. This frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even the space-like curve with a space-like binormal has vanishing second derivative [1]. He used tangent vector and any convenient arbitrary basis for the remainder of the frame. Then, the Bishop frame is expressed as [1]

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{2}$$

where

$$\begin{array}{c} \kappa(s) = \sqrt{|k_1^2 - k_2^2|}, \tau(s) = \dfrac{d\theta}{ds} \\ \theta(s) = \arg\tanh \dfrac{k_2}{k_1} \end{array}. \tag{3}$$

Here, we shall call k_1 and k_2 as Bishop curvatures.

§2. Main results

Let $\psi = \psi(s)$ be a space-like curve with a space-like binormal. We can write position vector with respect to Bishop frame as

$$\psi = \psi(s) = \gamma T + \delta N_1 + \lambda N_2. \tag{4}$$

Differentiating both sides of (4) and considering system (2), we have a system of differential equation as follows:

$$\begin{array}{r} \gamma' + \delta k_1 + \lambda k_2 - 1 = 0 \\ \delta' + \gamma k_1 = 0 \\ \lambda' - \gamma k_2 = 0 \end{array}. \tag{5}$$

This system of ordinary differential equations is a characterization for the curve $\psi = \psi(s)$. Position vector of a space-like curve can be determined by means of solution of it. However, general solution have not been found. Owing to this, we give some special values to components and Bishop curvatures.

Case I. $\gamma = 0$. In this case $\psi = \psi(s)$ lies fully in N_1N_2 subspace. Thus, we have other components

$$\begin{array}{l} \delta = \text{constant} = c_1 \\ \lambda = \text{constant} = c_2 \end{array}. \tag{6}$$

System (5) and (6) yield the following linear relation among Bishop curvatures

$$c_1k_1 + c_2k_2 - 1 = 0. \tag{7}$$

Since, we immediately arrive the following results.

Theorem 1. Let $\psi = \psi(s)$ be a space-like curve with a space-like binormal and lies fully in N_1N_2.

i) If one the second and third components of the position vector of ψ on Bishop axis is zero, then ψ transforms to a line.

ii) There is a relation among Bishop curvatures as (7).

iii) Position vector of ψ can be written as

$$\psi = \psi(s) = c_1N_1 + c_2N_2. \tag{8}$$

Case II. $\delta = \text{constant} \neq 0$. In this case, first we have by $(5)_2$ $\gamma = 0$ and $\lambda = \text{constant}$. Suffice it to say that this case is congruent to case I.

Case II. a. Let us suppose $\delta = 0$. Thus, $\gamma = 0$ and $\lambda = \text{constant}$, and so k_2 is constant. This case yields a line equation as follows

$$\psi = \psi(s) = \lambda N_2. \tag{9}$$

Case III. The case $\lambda = \text{constant} \neq 0$ is also congruent to case I.

Case III. a. $\lambda = 0$. Then, we easily have k_1 and δ are constants. This result follows a line equation as

$$\psi = \psi(s) = \delta N_1. \tag{10}$$

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Monotonicity and logarithmic convexity properties for the gamma function [1]

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Abstract We present the monotonicity and logarithmic convexity properties of the functions involving the gamma function. Minc-Sathre inequality are extended and refined.

Keywords Gamma function, monotonicity, logarithmically convexity, inequality.

§1. Introduction and results

The classical gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad (x > 0)$$

is one of the most important functions in analysis and its applications. The logarithmic derivative of the gamma function can be expressed in terms of the series

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{1+n} - \frac{1}{x+n} \right) \tag{1}$$

($x > 0$; $\gamma = 0.57721566490153286\cdots$ is the Euler's constant), which is known in literature as psi or digamma function. We conclude from (1) by differentiation

$$\psi^{(k)}(x) = (-1)^{k+1} k! \sum_{n=0}^{\infty} \frac{1}{(x+n)^{k+1}} \quad (x > 0;\ k = 1, 2, \cdots), \tag{2}$$

$\psi^{(k)}$ are called polygamma functions.

H. Minc and L. Sathre [1] proved that the inequality

$$\frac{n}{n+1} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < 1 \tag{3}$$

is valid for all natural numbers n. The inequality (3) can be refined and generalized as (see [2], [3], [4])

$$\frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i \right)^{1/n} \Bigg/ \left(\prod_{i=k+1}^{n+m+k} i \right)^{1/(n+m)} \leq \sqrt{\frac{n+k}{n+m+k}}, \tag{4}$$

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where k is a nonnegative integer, n and m are natural numbers. For $n = m = 1$, the equality in (4) is valid. The inequality (4) can be written as

$$\frac{n+k+1}{n+m+k+1} < \frac{[\Gamma(n+k+1)/\Gamma(k+1)]^{1/n}}{[\Gamma(n+m+k+1)/\Gamma(k+1)]^{1/(n+m)}} \leq \sqrt{\frac{n+k}{n+m+k}}. \tag{5}$$

In 1985, D. Kershaw and A. Laforgia [5] showed the function $[\Gamma(1+\frac{1}{x})]^x$ is strictly decreasing and $x[\Gamma(1+\frac{1}{x})]^x$ strictly increasing on $(0,\infty)$, from which the inequalities (3) can be derived. In 2003, B. -N. Guo and F. Qi [2] proved that the function

$$f(x) = \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1}$$

is decreasing in $x \geq 1$ for fixed $y \geq 0$, from which the left-hand side inequality of (5) can be obtained.

In this paper, our Theorem 1 considers the monotonicity and logarithmic convexity of the function f on $(0,\infty)$. This extends and generalizes Guo and Qi's result.

Theorem 1. Let $s \geq 0$ be real number, then the function

$$f(x) = \frac{[\Gamma(x+s+1)/\Gamma(s+1)]^{1/x}}{x+s+1}$$

is strictly decreasing and strictly logarithmically convex on $(0,\infty)$. Moreover,

$$\lim_{x\to 0} f(x) = e^{\psi(s+1)}/(s+1) \quad \text{and} \quad \lim_{x\to\infty} f(x) = e^{-1}.$$

Theorem 2. Let $s \geq 0$ be real number, then the function

$$g(x) = \frac{[\Gamma(x+s+1)/\Gamma(s+1)]^{1/x}}{\sqrt{x+s+1}}$$

is strictly increasing on $(0,\infty)$.

The following corollaries are obvious.

Corollary 1. Let $s \geq 0$ be a real number, then for all real numbers $x > 0$,

$$e^{-1} < \frac{[\Gamma(x+s+1)/\Gamma(s+1)]^{1/x}}{x+s+1} < \frac{e^{\psi(s+1)}}{(s+1)}. \tag{6}$$

Both bounds in (6) are best possible.

Corollary 2. Let $\alpha > 0$ and $s \geq 0$ be real numbers, then for all real numbers $x > 0$,

$$\frac{x+s+1}{x+\alpha+s+1} < \frac{[\Gamma(x+s+1)/\Gamma(s+1)]^{1/x}}{[\Gamma(x+\alpha+s+1)/\Gamma(s+1)]^{1/(x+\alpha)}} < \sqrt{\frac{x+s+1}{x+\alpha+s+1}}. \tag{7}$$

In particular, taking in (7) $x = n, s = 0$ and $\alpha = 1$, we obtain

$$\frac{n+1}{n+2} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < \sqrt{\frac{n+1}{n+2}}. \tag{8}$$

The inequality (8) is an improvement of (3).

§2. Proof of the theorems

Proof of Theorem 1. Define for $x > 0$,

$$
\begin{aligned}
u(x) &= x^2 \frac{f'(x)}{f(x)} = -\ln \frac{\Gamma(x+s+1)}{\Gamma(s+1)} + x\psi(x+s+1) - \frac{x^2}{x+s+1}, \\
v(x) &= x^3 \frac{d^2[\ln f(x)]}{dx^2} = 2\ln \frac{\Gamma(x+s+1)}{\Gamma(s+1)} - 2x\psi(x+s+1) \\
&\qquad + x^2\psi'(x+s+1) + \frac{x^3}{(x+s+1)^2}.
\end{aligned}
$$

Differentiation of $u(x)$ gives

$$
\begin{aligned}
\frac{1}{x}u'(x) &= \psi'(x+s+1) - \frac{1}{x+s+1} - \frac{s+1}{(x+s+1)^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{(x+s+n)^2} - \sum_{n=1}^{\infty} \left[\frac{1}{x+s+n} - \frac{1}{x+s+n+1}\right] \\
&\quad - \sum_{n=1}^{\infty} \left[\frac{s+1}{(x+s+n)^2} - \frac{s+1}{(x+s+n+1)^2}\right] \\
&= -\sum_{n=1}^{\infty} \left[\frac{s}{(x+s+n)^2} + \frac{1}{(x+s+n)(x+s+n+1)} - \frac{s+1}{(x+s+n+1)^2}\right] \\
&= -\sum_{n=1}^{\infty} \frac{(2s+1)(x+s+n)+s}{(x+s+n)^2(x+s+n+1)^2} < 0.
\end{aligned}
$$

Hence, the function $u(x)$ is strictly decreasing and $u(x) < u(0) = 0$ for $x > 0$, which yields the desired result that $f'(x) < 0$ for $x > 0$.

Using the asymptotic expansion [6, p. 257]

$$
\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \frac{1}{12x} + O(x^{-3}) \quad (x \to \infty),
$$

we conclude from

$$
\ln f(x) = \frac{1}{x}\left[\ln\Gamma(x+s+1) - \ln\Gamma(s+1)\right] - \ln(x+s+1) \tag{9}
$$

that

$$
\lim_{x\to\infty} f(x) = e^{-1}.
$$

By L' Hospital rule, we conclude from (9) that

$$
\lim_{x\to 0} f(x) = \frac{e^{\psi(s+1)}}{s+1}.
$$

Differentiation of $v(x)$ yields

$$\begin{aligned}\frac{1}{x^2}v'(x) &= \psi''(x+s+1)+\frac{1}{(x+s+1)^2}+\frac{2(s+1)}{(x+s+1)^3}\\ &= -\sum_{n=1}^{\infty}\frac{2}{(x+s+n)^3}+\sum_{n=1}^{\infty}\left[\frac{1}{(x+s+n)^2}-\frac{1}{(x+s+n+1)^2}\right]\\ &\quad+\sum_{n=1}^{\infty}\left[\frac{2(s+1)}{(x+s+n)^3}-\frac{2(s+1)}{(x+s+n+1)^3}\right]\\ &= \sum_{n=1}^{\infty}\frac{3(2s+1)(x+s+n)^2+(6s+1)(x+s+n)+2s}{(x+s+n)^3(x+s+n+1)^3}>0.\end{aligned}$$

Hence, the function $v(x)$ is strictly increasing and $v(x)>v(0)=0$ for $x>0$, which yields the desired result that $\frac{d^2[\ln f(x)]}{dx^2}>0$ for $x>0$.

Proof of Theorem 2. Define for $x>0$,

$$p\,(x)=x^2\frac{g'(x)}{g(x)}=-\ln\frac{\Gamma(x+s+1)}{\Gamma(s+1)}+x\psi(x+s+1)-\frac{x^2}{2(x+s+1)}.$$

Differentiation of $p\,(x)$ gives

$$\begin{aligned}\frac{1}{x}p'(x) &= \psi'(x+s+1)-\frac{1}{2(x+s+1)}-\frac{s+1}{2(x+s+1)^2}\\ &= \sum_{n=1}^{\infty}\frac{1}{(x+s+n)^2}-\frac{1}{2(x+s+1)}-\frac{s+1}{2(x+s+1)^2}\\ &> \int_1^{\infty}\frac{dt}{(x+s+1)^2}-\frac{1}{2(x+s+1)}-\frac{s+1}{2(x+s+1)^2}\\ &= \frac{x}{2(x+s+1)^2}>0.\end{aligned}$$

Hence, the function $p\,(x)$ is strictly increasing and $p\,(x)>p\,(0)=0$ for $x>0$, which yields the desired result that $g'(x)>0$ for $x>0$.

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On ideas and congruences in KU-algebras

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Abstract In this paper, we will introduce some kind of algebras which is called KU-algebras. We define ideals and study congruences on KU-algebras, and also investigate some related properties.

Keywords ideals, congruences, KU-algebras

§1. Preliminaries

Several authors ([1],[2],[3],[4],[5]) introduced some structures of algebras such as BCK, BCC and BCI-algebras. In [2], C.S. Hoo introduced the notions of filters and commutative ideals in BCI-algebras. [1] and [3] introduced a concept of ideals in BCC and BCK-algebras. W. A. Dudek and X. Zhang gave connections between ideals and congruences in BCC-algebras. The objective of this paper is to introduce KU-algebras and also study ideals and congruences in KU-algebras. Moreover, we investigate some of its properties.

By an algebra $G = (G, \cdot, 0)$ we mean a non-empty set G together with a binary operation, multiplication and a some distinguished element 0. In the sequel a multiplication will be denoted by juxtaposition.

Definition 1. An algebra $G = (G, \cdot, 0)$ is called a KU-algebra if it satisfies the following conditions:

(1) $(xy)((yz)(xz)) = 0$

(2) $0x = x$,

(3) $x0 = 0$,

(4) $xy = 0 = yx$ implies $x = y$

for all $x, y, z \in G$.

By (1), we get $(00)((0x)(0x)) = 0$. It follows that $xx = 0$ for all $x \in G$. And if we put $y = 0$ in (1), then we obtain $z(xz) = 0$ for all $x, z \in G$.

Example 1. Let $G = \{0, 1, 2, 3\}$ and $H = \{0, 1, 2, 3, 4\}$. Let the multiplication of G and H be defined by Table 1 and Table 2 respectively. Then it is easily checked that G and H are KU-algebras.

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	0	2
2	0	2	0	1
3	0	0	0	0

Table 1.

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	3	4
2	0	1	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Table 2.

§2. Ideals

Definition 2. Let A be a non-empty subset of a KU-algebra G. Then A is said to be an ideal of G if it satisfies the following conditions:

(i) $0 \in A$,

(ii) for all $x, y, z \in G$, $x(yz) \in A$ and $y \in A$ imply $xz \in A$.

Example 2. In example 1, let $A = \{0, 2\}$ and $B = \{0, 1\}$ be subsets of a KU-algebra H. Then A is an ideal of H but B is not an ideal of H.

Putting $x = 0$ in Definition 2 we obtain the following Proposition:

Proposition 1. Let A be an ideal of a KU-algebra G. Then for all $x, y \in G$, $xy \in A$ and $x \in A$ imply $y \in A$.

Definition 3. Let $(G, \cdot, 0)$ be a KU-algebra. Then a non-empty subset S of G is said to be a KU-subalgebra of G if $(S, \cdot, 0)$ is KU-algebra.

Note that S is a KU-subalgebra of G if and only if $xy \in S$ for all $x, y \in S$.

Proposition 2. Let A be an ideal of KU-algebra G. Then A is a KU-subalgebra of G.

Proof of Proposition 2. Let $x, y \in A$. Then $(x0)((0y)(xy)) = 0$. Hence $y(xy) \in A$. Since A is an ideal of G and $y \in A$, $xy \in A$. Therefore, A is a KU-algebra of G.

Proposition 3. Let G be a KU-algebra and A a non-empty subset of G with 0. Then A is an ideal of G if and only if $x \in A$, $yz \notin A$ imply $y(xz) \notin A$ for all $x, y, z \in G$.

Proof of Proposition 3. Let A be an ideal of G and let $x \in A$, $yz \notin A$. Suppose that $y(xz) \in A$. Since A is an ideal, $yz \in A$, a contradiction.

Conversely, assume that $x \in A$, $yz \notin A$ imply $y(xz) \notin A$ for all $x, y, z \in G$. Let $x, y, z \in G$ be such that $x(yz) \in A$ and $y \in A$. It is clear that $xz \in A$.

Corollary 1. Let G be a KU-algebra and A a non-empty subset of G with 0. Then A is an ideal of G if and only if $x \in A$, $y \notin A$ imply $yx \notin A$ for all $x, y \in G$.

On KU-algebra $(G, \cdot, 0)$. We define a binary relation $\leq$ on G by putting $x \leq y$ if and only if $yx = 0$. Then $(G; \leq)$ is a partially ordered set and 0 is its smallest element. Thus a KU-algebra G satisfies conditions: $(yz)(xz) \leq xy$, $0 \leq x$, $x \leq y \leq x$ implies $x = y$.

§3. Congruences

In this topic, we describe congruences on KU-algebras. We start with the following:

Definition 4. Let A be an ideal of KU-algebra G. Define the relation $\sim$ on G by

$$x \sim y \text{ iff } xy \in A \text{ and } yx \in A$$

Theorem 1. If A is an ideal of KU-algebra G, then the relation $\sim$ is a congruence on G.

Proof of Theorem 1. It is clear that this relation is reflexive and symmetric. Let $x, y, z \in G$ be such that $x \sim y$ and $y \sim z$. Then $xy, yx, yz, zy \in A$ and $(xy)((yz)(xz)) = 0 \in A$. By Proposition 1, $xz \in A$. Similarly, $(zy)((yx)(zx) = 0 \in A$. Thus $x \sim z$. Therefore, $\sim$ is an equivalenec relation.

If $x \sim u$ and $y \sim v$ for $x, y, u, v \in G$, then $xu, ux, yv, vy \in A$ and $(xu)((uy)(xy)) = 0 \in A$. By Proposition 1, $(uy)(xy) \in A$. Similarly $(xy)(uy) \in A$. Thus $xy \sim uy$. On the other hand $(uy)((yv)(uy)) = 0 \in A$ and $yv \in A$ imply $(uy)(uv) \in A$. Similarly, if $(uv)((vy)(uy)) = 0 \in A$ and $vy \in A$ we obtain $(uv)(uy) \in A$. Thus $uy \sim uv$. Since $\sim$ is transitive, $xy \sim uv$. Hence $\sim$ is a congruence.

Proposition 4. If $\sim$ is a congruence on a KU-algebra G, then $C_0 = \{x \in G \mid x \sim 0\}$ is an ideal of G.

Proof of Proposition 4. Obviously $0 \in C_0$. If $x(yz) \in C_0$ and $y \in C_0$, then $x(yz) \sim 0$ and $y \sim 0$. Since $x \sim x$ and $z \sim z$, $x(yz) \sim x(0z)$. Thus $xz \sim 0$. Hence $xz \in C_0$. So C_0 is an ideal of G.

Let $\sim$ be a congruence relation on a KU-algebra G and let $C_x = \{y \in G \mid y \sim x\}$. Then the family $\{C_x : x \in G\}$ gives a partition of G which is denoted by $G/_\sim$. For any $x, y \in G$, we define $C_x * C_y = C_{xy}$. Since $\sim$ has the substitution property, the operation * is well-defined. It is easily checked that $(G/_\sim, *, C_0)$ is a KU-algebra.

Definition 5. Let $(G, \cdot, 0)$ and $(H, *, 0)$ be KU-algebras. A homomorphism is a map $f : G \to H$ satisfying $f(x \cdot y) = f(x) * f(y)$ for all $x, y \in G$. An injective homomorphism is called monomorphism and a surjective homomorpism is called epimorphism.

The kernel of the homomorphism f, denoted by ker$f$, is the set of elements of G that map to 0 in H.

If f is a homomorphism from a KU-algebra G into a KU-algebra H, then we define kernel ker$f = f^{-1}(0)$ as in [1] and we can prove the following result:

Theorem 2. Let G be a KU-algebra and A an ideal of G. Let $G/_\sim$ be a KU-algebra determined by A. Then the canonical mapping $f : G \to G_- \sim$ defined by $f(x) = C_x$ is an epimorphism and ker$f$ is an ideal of G.

Proof of Theorem 2. Let $x, y \in G$. If $x = y$, then $C_x = C_y$. Thus $f(x) = f(y)$. Since $f(xy) = C_{xy} = C_x * C_y = f(x) * f(y)$, f is a homomorphism. Let $C_x \in G/_\sim$. We get $f(x) = C_x$. That is f is an epimomorphism.

Since $f(0) = C_0$, ker$f \neq \phi$. If $x(yz) \in$ker$f$ and $y \in$ker$f$. Then $f(x(yz)) = f(y) = C_0$. Thus $C_0 = f(x) * (f(y) * f(z)) = f(x) * (C_0 * f(z)) = f(x) * f(z) = f(xz)$. Thus $xy \in$ker$f$. It follows that ker$f$ is an ideal of G.

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Browder's theorem and generalized Weyl's theorem

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Abstract Two variants of the Weyl spectrum are discussed; We find for example that if one of them coincides with the Drazin spectrum then generalized Weyl's theorem holds, and conversely for isoloid operators.

Keywords Browder's theorem, generalized Weyl's theorem, operators.

§1. Introduction

H.Weyl [22] examined the spectra of all compact perturbations of a hermitian operator on Hilbert space and found in 1909 that their intersection consisted precisely of those points of the spectrum which were not isolated eigenvalues of finite multiplicity. This " Weyl's theorem " has since been extended to hyponormal and to Toeplitz operators (Coburn [8]), to seminormal and other operators (Berberian [2], [3]) and to Banach spaces operators (Istrătescu [11], Oberai [17]). Variants have been discussed by Harte and Lee [10] and Rakočevič [18], M.Berkani and J.J.Koliha [6]. In this note we show how generalized Weyl's theorem follows from the equality of the Drazin spectrum and a variant of the Weyl's spectrum.

Recall that the " Weyl's spectrum " of a bounded linear operator T on a Banach space X is the intersection of the spectra of its compact perturbations:

$$\sigma_w(T) = \bigcap\{\sigma(T+K) : K \in K(X)\} \ . \tag{1}$$

Equivalently $\lambda \in \sigma_w(T)$ iff $T-\lambda I$ fails to be Fredholm of index zero. The " Browder spectrum " is the intersection of the spectra of its commuting compact perturbations:

$$\sigma_b(T) = \bigcap\{\sigma(T+K) : K \in K(X) \cap \mathrm{comm}(T)\} \ . \tag{2}$$

Equivalently $\lambda \in \sigma_b(T)$ iff $T-\lambda I$ fails to be Fredholm of finite ascent and descent. " the Weyl's theorem holds " for T iff

$$\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T) \ , \tag{3}$$

where we write

$$\pi_{00}(T) = \{\lambda \in \mathrm{iso}\ \sigma(T) : \ 0 < \dim N(T-\lambda I) < \infty\} \tag{4}$$

for the isolated points of the spectrum which are eigenvalues of finite multiplicity. Harte and Lee [10] have discussed a variant of Weyl's theorem: " the Browder's theorem holds " for T iff

$$\sigma(T) = \sigma_w(T) \cup \pi_{00}(T) \ . \tag{5}$$

What is missing is the disjointness between the Weyl spectrum and the isolated eigenvalues of finite multiplicity: equivalently

$$\sigma_w(T) = \sigma_b(T) \ . \tag{6}$$

For a bounded linear operator T and a nonnegative integer n define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range space $R(T^n)$ is closed and $T_{[n]}$ is upper (resp.a lower) semi-Fredholm operator, then T is called an upper (resp.lower) semi-B-Fredholm operator. Moreover if $T_{[n]}$ is a Fredholm(Weyl or Browder) operator, then T is called a B-Fredholm (B-Weyl or B-Browder) operator. Similarly, we can define the upper semi-B-Fredholm, B-Fredholm, B-Weyl, and B-Browder spectrums $\sigma_{SF_+}(T)$, $\sigma_{BF}(T)$, $\sigma_{BW}(T)$, $\sigma_{BB}(T)$. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator.

(See [14]) Let $T \in B(X)$ and let

$$\triangle(T) = \{n \in \mathbb{N} : \forall m \in \mathbb{N}, m \geq n \Rightarrow [R(T^n) \cap N(T)] \subseteq [R(T^m) \cap N(T)]\}.$$

Then the degree of stable iteration dis(T) of T is defined as $dis(T) = inf \triangle (T)$.

Let T be a semi-B-Fredholm operator and let d be the degree of the stable iteration of T. It follows from [4, Proposition 2.1] that if $T_{[d]}$ is a semi-Fredholm operator, and $ind(T_{[m]}) = ind(T_{[d]})$ for each $m \geq d$. This enables us to define the index of a semi-B-Fredholm operator T as the index of the semi-Fredholm operator $T_{[d]}$.

In the case of a normal operator T acting on a Hilbert space, Berkani [5, Theorem 4.5] showed that

$$\sigma_{BW}(T) = \sigma(T) \backslash E(T),$$

$E(T)$ is the set of all eigenvalues of T which are isolated in the spectrum of T. This result gives a generalization of the classical Weyl's theorem. We say T obeys generalized Weyl's theorem if $\sigma_{BW}(T) = \sigma(T) \backslash E(T)$([6, Definition 2.13]).

In this paper, we describe Browder's theorem and generalized Weyl's theorem using two new spectrum sets which we define in section 2.

§2. Browder's theorem and generalized Weyl's theorem

Using Corollary 4.9 in [9], we can say that $\sigma_{BB}(T) = \sigma_D(T)$, where $\sigma_D(T) = \{\lambda \in \sigma(T) : \lambda$ is not a pole of T $\}$. We call $\sigma_D(T)$ the Drazin spectrum of T. We can prove that the Drazin spectrum satisfies the spectral mapping theorem, and the Drazin spectrum of a direct sum is the union of the Drazin spectrum of the components.

In this section, our first result is:

Theorem 2.1. Browder's theorem holds for T if and only if $\sigma_{BW}(T) = \sigma_D(T)$.

Proof. Suppose that Browder's theorem holds for T. We only need to prove that $\sigma_D(T) \subseteq \sigma_{BW}(T)$. If λ_0 is not in $\sigma_{BW}(T)$, then $T - \lambda_0 I$ is B-Weyl operator, and, in particular, an operator of topological uniform descent. [7, Remark iii] asserts that there exists $\epsilon > 0$ such that $T - \lambda I$

is Weyl if $0 < |\lambda - \lambda_0| < \epsilon$. Since Browder's theorem holds for T, it follows that $T - \lambda I$ is Browder operator if $0 < |\lambda - \lambda_0| < \epsilon$. Then λ_0 is a boundary of $\sigma(T)$. [9, Corollary 4.9.] tells us that λ_0 is not in $\sigma_D(T)$. Conversely, Suppose that $\sigma_{BW}(T) = \sigma_D(T)$. We need to prove that $\sigma_b(T) = \sigma_w(T)$. Suppose that $T - \lambda_0 I$ is Weyl. Then λ_0 is not in $\sigma_D(T)$, which means that $\lambda_0 \in iso\sigma(T)$. Thus $T - \lambda_0 I$ is Browder. This proves that Browder's theorem holds for T.

Theorem 2.2. If Browder's theorem holds for $T \in B(X)$ and $S \in B(X)$, and p is a polynomial, then Browder's theorem holds for

$$p(T) \Longleftrightarrow p(\sigma_{BW}(T)) = \sigma_{BW}(p(T));$$

Browder's theorem holds for

$$T \oplus S \Longleftrightarrow \sigma_{BW}(T \oplus S) = \sigma_{BW}(T) \cup \sigma_{BW}(S).$$

Proof. Browder's theorem holds for $p(T)$ if and only if $\sigma_{BW}(p(T)) = \sigma_D(p(T)) = p(\sigma_D(T)) = p(\sigma_{BW}(T))$ and Browder's theorem holds for $T \oplus S$ if and only if $\sigma_{BW}(T \oplus S) = \sigma_D(T \oplus S) = \sigma_D(T) \cup \sigma_D(S) = \sigma_{BW}(T) \cup \sigma_{BW}(S)$.

Theorem 2.3. If $T \in B(X)$, then

$$ind(T - \lambda I)ind(T - \mu I) \geq 0 \text{ for each } \text{pair}\lambda, \mu \in \mathbb{C}\backslash\sigma_e(T)$$

if and only if

$$p(\sigma_{BW}(T)) = \sigma_{BW}(p(T)) \text{ for each polynomial } p.$$

Proof. By [7, Remark iii], $ind(T - \lambda I)ind(T - \mu I) \geq 0$ *for each pair*$\lambda, \mu \in \mathbb{C}\backslash\sigma_e(T)$ if and only if $ind(T - \lambda I)ind(T - \mu I) \geq 0$ for each pair $\lambda, \mu \in \mathbb{C}\backslash\sigma_{BF}(T)$. From [7, Corollary 3.3] and [5, Theorem 3.2], the spectral mapping theorem for the B-Weyl spectrum may be rewritten as the implication, for arbitrary $n \in \mathbb{N}$ and $\lambda_i \in \mathbb{C}$,

$$(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I) \text{ B} - \text{Weyl} \Longrightarrow T - \lambda_j I \text{ B} - \text{Weyl for each } j = 1, 2, \cdots, n.$$

Now if $ind(T - \lambda I) \geq 0$ on $\mathbb{C}\backslash\sigma_{BF}(T)$, then we have

$$\sum_{j=1}^{n} ind(T - \lambda_j I) = ind \prod_{j=1}^{n}(T - \lambda_j I) = 0 \Longrightarrow ind(T - \lambda_j I) = 0 (j = 1, 2, \cdots, n),$$

and similarly if $ind(T - \lambda I) \leq 0$ on $\mathbb{C}\backslash\sigma_{BF}(T)$. If conversely there exist $\lambda, \mu \in \mathbb{C}\backslash\sigma_e(T)$ for which $ind(T - \lambda I) = -m < 0 < k = ind(T - \mu I)$, then $p(T) = (T - \lambda I)^k (T - \mu I)^m$ is a Weyl operator whose factors are not B-Weyl. It is a contradiction. The proof is completed.

We turn to a variant of the Weyl spectrum, involving a condition introduced by Saphar [20] and the " zero jump " condition of Kato [12]. Let:

$$\rho_1(T) = \{\lambda \in \mathbb{C} : \text{there exists } \epsilon > 0 \text{ such that } T - \mu I \text{ is Weyl and}$$

$$N(T - \mu I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \mu I)^n] \text{ if } 0 < |\mu - \lambda| < \epsilon\}$$

and let $\rho_D(T) = \mathbb{C}\backslash\sigma_D(T)$, $\rho(T) = \mathbb{C}\backslash\sigma(T)$, $\sigma_1(T) = \mathbb{C}\backslash\rho_1(T)$. Then $\rho(T) \subseteq \rho_D(T) \subseteq \rho_1(T)$.

T is called isoloid if $\lambda \in iso\sigma(T) \Longrightarrow N(T-\lambda I) \neq \{0\}$.

Theorem 2.4. $T \in B(X)$ is isoloid and generalized Weyl's theorem holds for T if and only if $\sigma_1(T) = \sigma_D(T)$.

Proof. Suppose T is isoloid and generalized Weyl's theorem holds for T. We only need to prove $\sigma_D(T) \subseteq \sigma_1(T)$. Let $\lambda_0 \in \sigma_D(T)$ and suppose $\lambda_0 \in \rho_1(T)$. Then there exists $\epsilon > 0$ such that $T-\lambda I$ is Weyl and $N(T-\lambda I) \subseteq \bigcap\limits_{n=1}^{\infty} R[(T-\lambda I)^n]$ if $0 < |\lambda - \lambda_0| < \epsilon$. Since generalized Weyl's theorem implies Weyl's theorem for T ([6, Theorem 3.9]), it follows that $T-\lambda I$ is Browder and therefor $N(T-\lambda I) = N(T-\lambda I) \cap \bigcap\limits_{n=1}^{\infty} R[(T-\lambda I)^n] = \{0\}$ for $0 < |\lambda - \lambda_0| < \epsilon$, which means that $T-\lambda I$ is invertible if $0 < |\lambda-\lambda_0| < \epsilon$. Then λ_0 is an isolated point in $\sigma(T)$. Thus $\lambda_0 \in E(T) = \sigma(T)\backslash\sigma_{BW}(T)$ because T is isoloid. [9, Corollary 4.9] asserts that λ_0 is not in $\sigma_D(T)$, it is a contradiction.

Conversely, Suppose $\sigma_1(T) = \sigma_D(T)$. By $E(T) \subseteq \rho_1(T) = \rho_D(T)$, we get $E(T) \subseteq \sigma(T)\backslash\sigma_{BW}(T)$. Conversely, let $\lambda_0 \in \sigma(T)\backslash\sigma_{BW}(T)$, that is $T-\lambda_0 I$ is B-Weyl, then there exists $\epsilon > 0$ such that $T-\lambda I$ is Weyl and $N(T-\lambda I) \subseteq \bigcap\limits_{n=1}^{\infty} R[(T-\lambda I)^n]$ if $0 < |\lambda-\lambda_0| < \epsilon$, then $\lambda_0 \in \rho_1(T) = \rho_D(T)$. Thus $\lambda_0 \in E(T)$. In the following, we will prove T is isoloid. Let $\lambda_0 \in iso\sigma(T)$, then $\lambda_0 \in \rho_1(T) = \rho_D(T)$, thus λ_0 is a pole of T, so $N(T-\lambda_0 I) \neq \{0\}$, which means that T is isoloid.

Corollary 2.5. Suppose $T, S \in B(X)$ are all isoloid. If generalized Weyl's theorem holds for T and S and if p is a polynomial, then generalized Weyl's theorem holds for

$$p(T) \Longleftrightarrow \sigma_1(p(T)) = p(\sigma_1(T))$$

and generalized Weyl's theorem holds for

$$T \oplus S \Longleftrightarrow \sigma_1(T \oplus S) = \sigma_1(T) \cup \sigma_1(S).$$

Proof. If T and S are isoloid, then $p(T)$ and $T \oplus S$ are isoloid. Then generalized Weyl's theorem holds for $p(T) \Longleftrightarrow \sigma_1(p(T)) = \sigma_D(p(T)) = p(\sigma_D(T)) = p(\sigma_1(T))$ and generalized Weyl's theorem holds for $T \oplus S \Longleftrightarrow \sigma_1(T\oplus S) = \sigma_D(T \oplus S) = \sigma_D(T) \cup \sigma_D(S) = \sigma_1(T) \cup \sigma_1(S)$.

In the following, we suppose that $H(T)$ ($H(\sigma(T))$) is the class of all complex-valued functions which are analytic on a neighborhood (region) of $\sigma(T)$.

Theorem 2.6. $T \in B(X)$, then

$$ind(T-\lambda I)ind(T-\mu I) \geq 0 \text{ for each pair} \lambda, \mu \in \mathbb{C}\backslash\sigma_e(T)$$

if and only if

$$f(\sigma_1(T)) \subseteq \sigma_1(f(T)) \text{ for } \text{ any} f \in H(T).$$

Proof. Suppose $ind(T-\lambda I)ind(T-\mu I) \geq 0$ for each pair $\lambda, \mu \in \mathbb{C}\backslash\sigma_e(T)$. For any $f \in H(T)$, let $\mu_0 \in f(\sigma_1(T))$ and suppose $\mu_0 = f(\lambda_0)$, where $\lambda_0 \in \sigma_1(T)$. If μ_0 is not in $\sigma_1(f(T))$, then there exists $\delta > 0$ such that $f(T)-\mu I$ is Weyl and $N(f(T)-\mu I) \subseteq \bigcap\limits_{n=1}^{\infty} R[(f(T)-\mu I)^n]$

if $0 < |\mu - \mu_0| < \delta$. Then $\mu \in \rho_k(f(T))$, where $\rho_k(f(T)) = \{\lambda \in \mathbb{C} : R(f(T) - \lambda I)$ is closed and $N(f(T) - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(f(T) - \lambda I)^n]\}$, which is defined by T.Kato in [14]. In the following we will prove that $\lambda_0 \in \rho_1(T)$. By continuity of $f(\lambda)$, there exists $\epsilon > 0$ such that $0 < |f(\lambda_0') - f(\lambda_0)| = |f(\lambda_0') - \mu_0| < \delta$ if $0 < |\lambda_0' - \lambda_0| < \epsilon$. Then $f(T) - f(\lambda_0')I$ is Weyl and $N(f(T) - f(\lambda_0')I) \subseteq \bigcap_{n=1}^{\infty} R[(f(T) - f(\lambda_0')I)^n]$. Thus $f(\lambda_0')$ is not in $\sigma_k(f(T)) = f(\sigma_k(T))$ ([21, Satz 6]). So $\lambda_0' \in \rho_k(T)$, which means that $N(T - \lambda_0' I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda_0' I)^n]$. Clearly, λ_0 is not an isolated point of $\sigma(T)$. Suppose $\lambda_0' \in \sigma(T)$ satisfies $0 < |\lambda_0' - \lambda_0| < \epsilon$. Let $h(\lambda) = f(\lambda) - f(\lambda_0')$. Then $h(\lambda) \neq 0$ for all $\lambda \in \sigma_k(T)$. Clearly, h has zeros in $\sigma(T)$. [21, Satz 3] asserts now that h has only a finite number of zeros in $\sigma(T)$. Let $\lambda_0', \lambda_1', \cdots, \lambda_m'$ be these zeros ($\lambda_i \neq \lambda_j$ for $i \neq j$) and $n_0, n_1, n_2, \cdots, n_m$ be their respective orders. Then we can denote $f(T) - f(\lambda_0')I$ by

$$f(T) - f(\lambda_0')I = (T - \lambda_0' I)^{n_0} (T - \lambda_1' I)^{n_1} \cdots (T - \lambda_m' I)^{n_m} g(T) ,$$

where $g(T)$ is invertible and $\lambda_i \neq \lambda_j$ for $i, j = 0, 1, 2, \cdots, m$. Since $f(T) - f(\lambda_0')I$ is Weyl, it follows that $T - \lambda_i' I$ is Fredholm and $0 = ind[f(T) - f(\lambda_0')I] = \sum_{i=0}^{m} ind(T - \lambda_i' I)^{n_i}$. Thus $T - \lambda_i' I$ is Weyl. We now get that there exists $\epsilon > 0$ such that $T - \lambda_0' I$ is Weyl and $N(T - \lambda_0' I) \subseteq \bigcap_{n=1}^{\infty} R[(T - \lambda_0' I)^n]$ if $0 < |\lambda_0' - \lambda_0| < \epsilon$. Then $\lambda_0 \in \rho_1(T)$. It is in contradiction to the fact $\lambda_0 \in \sigma_1(T)$. Then $f(\sigma_1(T)) \subseteq \sigma_1(f(T))$ for any $f \in H(T)$.

For the converse, if there exist $\lambda, \mu \in \mathbb{C} \backslash \sigma_e(T)$ for which $ind(T - \lambda I) = -m < 0 < k = ind(T - \mu I)$, let $f(T) = (T - \lambda I)^k (T - \mu I)^m$. Then $0 \in f(\sigma_1(T))$ but 0 is not in $\sigma_1(f(T))$. It is a contradiction. The proof is completed.

Corollary 2.7. If $T \in B(X)$ is isoloid and generalized Weyl's theorem holds for T, then the following statements are equivalent:

(1) $ind(T - \lambda I) ind(T - \mu I) \geq 0$ for each pair $\lambda, \mu \in \mathbb{C} \backslash \sigma_e(T)$;

(2) $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ for every $f \in H(\sigma(T))$;

(3) generalized Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$;

(4) $\sigma_1(f(T)) = f(\sigma_1(T))$ for every $f \in H(\sigma(T))$.

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On the instantaneous screw axes of two parameter motions in Lorentzian space

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Abstract In this study two parameter motion is given by using the rank of rotation matrix in Lorentzian space. It is shown locus of instantaneous screw axis is a ruled surface at any position of $(\lambda,\, \mu) = (0, 0)$.

Keywords Two parameter motion, one parameter motion, instantaneous screw axis (I.S.A), Lorentzian Space.

§1. Introduction and preliminaries

Let $IR^n = \{(r_1, r_2, \cdots, r_n) \mid r_1, r_2, ..., r_n \in IR\}$ be a $n-$dimensional vector space, $r = (r_1, r_2, \cdots, r_n)$ and $s = (s_1, s_2, \cdots, s_n)$ be two vectors in IR^n, the Lorentz scalar product of r and s is defined by

$$\langle r, s\rangle_L = -r_1 s_1 + r_2 s_2 + \cdots + r_n s_n \quad .$$

$L^n = (IR^n, \langle,\rangle_L)$ is called $n-$dimensional Lorentz space, or Minkowski $n-$space. We denote L^nas $(IR^n, \langle,\rangle_L)$. For any $r = (r_1, r_2, r_3)$, $s = (s_1, s_2, s_3) \in L^3$, in the meaning Lorentz vector product of r and s is defined by

$$r \wedge_L s = (r_2 s_3 - r_3 s_2, r_1 s_3 - r_3 s_1, r_2 s_1 - r_1 s_2)\,,$$

where $e_1 \wedge_L e_2 = -e_3$, $e_2 \wedge_L e_3 = e_1$vector, a lightlike vector or a timelike vector if $\langle r, r\rangle_L > 0$, $\langle r, r\rangle_L = 0$ or $\langle r, r\rangle_L < 0$ respectively. For $r \in L^n$, the norm of r defined by $\|r\|_L = \sqrt{|\langle r, r\rangle_L|}$, and r is called a unit vector if $\|r\|_L = 1$ [5]. In the Minkowski n-space, the two parameter motion of a rigid body is defined by

$$Y(\lambda, \mu) = A(\lambda, \mu)X + C(\lambda, \mu), \tag{1.1}$$

where $A \in SO(n, 1)$ is a positive semi orthogonal matrix, $C \in IR^n_1$ is a column matrix, Y and X are position vectories of the same point B respectively, for the fixed and moving space with respect to semi orthonormal coordinate systems. The two parameter motion is given by (1.1), for $(\lambda, \mu) = (0, 0)$, we have

$$A(0, 0) = A^{-1}(0, 0) = A^T(0, 0) = I$$

and

$$C(0,0) = 0.$$

Then fixed and moving space is coincided. If $\lambda = \lambda(t), \mu = \mu(t)$, then one parameter motion is obtained from two parameter motion. Since $A \in SO(n,1)$, we have

$$A^T(\lambda,\mu)\varepsilon A(\lambda,\mu)\varepsilon = A(\lambda,\mu)\varepsilon A^T(\lambda,\mu)\varepsilon = I_n, \text{where } \varepsilon = \begin{bmatrix} -1 & 0 & . & 0 & 0 \\ 0 & 1 & . & . & . \\ . & . & . & . & . \\ 0 & . & . & . & 0 \\ 0 & . & . & 0 & 1 \end{bmatrix}_{nxn}.$$

For the sake of the short we shall take as $A^T(\lambda,\mu) = A^T$ and $A(\lambda,\mu) = A$.

Definition 1. Taking the derivation with respect to t in equation $Y(\lambda,\mu) = A(\lambda,\mu)X + C(\lambda,\mu)$ where let $\lambda = \lambda(t)$ and $\mu = \mu(t)$, then it follows that

$$\dot{Y} = Y_\lambda\dot{\lambda} + Y_\mu\dot{\mu},$$

$$\dot{A} = A_\lambda\dot{\lambda} + A_\mu\dot{\mu},$$

$$\dot{C} = C_\lambda\dot{\lambda} + C_\mu\dot{\mu},$$

$$\dot{Y} = \dot{A}X + \dot{C} + A\dot{X}.$$

So $\dot{Y}$, $\dot{A}X + \dot{C}$, $A\dot{X}$ are called absolutely sliding and relative velocities of the point B has position vectories $\overrightarrow{b}$, respectively. Let X be solution of system of $\overrightarrow{V_f} = \dot{A}X + \dot{C} = 0$ and the solution is constant on the fixed and moving space at position t. These points X is called instantaneous pole points at every position t.

Definition 2. If $rank\,\dot{A} = n - 1 = r$ be an even number on the two parameter motion given by equation $Y(\lambda,\mu) = A(\lambda,\mu)X + C(\lambda,\mu)$, then at any position of points the locus having a velocity vector with stationary norm is a line. The line is called instantaneous screw axis and denoted by I.S.A. [2]. Furthermore the moving space screw axis is defined by $X = P + \sigma E$ where P is a particular solution of equation $\dot{A}X + \dot{C} = 0$ and E represent a bases of solution space of homogeneous equation $\dot{A}X = 0$.

§2. The instantaneous screw axes of two parameter motions

Theorem 1. Let $A \in SO(n,1)$ and let n be an odd number. Then the rank of A_λ and A_μ are even.

Proof. Since

$$A^T(\lambda,\mu)\varepsilon A(\lambda,\mu)\varepsilon = A(\lambda,\mu)\varepsilon A^T(\lambda,\mu)\varepsilon = I_n$$

and $A(0,0) = A^T(0,0) = I$, then

$$A_\lambda\varepsilon A^T\varepsilon + A\varepsilon A_\lambda^T\varepsilon = 0, \varepsilon^2 = I_n$$

$$A_\lambda + \varepsilon A_\lambda^T \varepsilon = 0$$

and

$$A_\mu \varepsilon A^T \varepsilon + A \varepsilon A_\mu^T \varepsilon = 0$$

$$A_\mu + \varepsilon A_\mu^T \varepsilon = 0.$$

Thus A_λ and A_μ are semi skew-symmetric matrices. Since n is an odd number it follows that

$$\begin{aligned} \det A_\lambda &= 0, \\ \det A_\mu &= 0. \end{aligned}$$

Thus it must be $rank\ (A_\lambda) = r\,(\text{even})$ and $rank\,(A_\mu)\ = r\,(\text{even})$.

Theorem 2. Let $A \in SO(n,1)$. Then

$$rank\, A_{\lambda\lambda} = 0 \quad \Leftrightarrow \quad rank\, A_\lambda = 0$$

and

$$rank\, A_{\mu\mu} = 0 \quad \Leftrightarrow \quad rank\, A_\mu = 0.$$

Proof. Since

$$A(\lambda,\mu)\varepsilon A^T(\lambda,\mu)\varepsilon = I_n \tag{2.1}$$

it takes derivation with respect to λ, it follows that

$$A_\lambda \varepsilon A^T \varepsilon + A \varepsilon A_\lambda^T \varepsilon = 0$$

$$A_{\lambda\lambda} \varepsilon A^T \varepsilon + A_\lambda \varepsilon A_\lambda^T \varepsilon + A_\lambda \varepsilon A_\lambda^T \varepsilon + A \varepsilon A_{\lambda\lambda}^T \varepsilon = 0$$

$$A_{\lambda\lambda} A^T + 2A_\lambda \varepsilon A_\lambda^T \varepsilon + A \varepsilon A_{\lambda\lambda}^T \varepsilon = 0. \tag{2.2}$$

Since $rank\, A_{\lambda\lambda} = 0$, we get $A_{\lambda\lambda} = 0$ and $A_{\lambda\lambda}^T = 0$. We have following that (2.1)

$$A_\lambda \varepsilon A_\lambda^T \varepsilon = 0 \Rightarrow A_\lambda \varepsilon A_\lambda^T = 0.$$

For every $x \in IR_n^1$, we have

$$\left(A_\lambda \varepsilon A_\lambda^T\right)\, x^T = (0)\, x^T$$

$$\left(A_\lambda \varepsilon A_\lambda^T\right)\, x^T = 0$$

$$x\, \left(A_\lambda \varepsilon A_\lambda^T\right)\, x^T = 0$$

$$(x A_\lambda)\, \varepsilon\, (x A_\mu)^T = 0.$$

So that $\langle x A_\lambda,\ x A_\lambda \rangle_L = 0$ (from the non-degenere property), $x A_\lambda = 0$. Since it is true for every $x \in IR_n^1$, so we get $A_\lambda = 0$ and $rank\,(A_\lambda)\ = 0$. It takes derivation with respect to μ in the equation (2.1) similarly it follows that

$$A_\mu \varepsilon A^T \varepsilon + A \varepsilon A_\mu^T \varepsilon = 0,$$

$$A_{\mu\mu} \varepsilon A^T \varepsilon + A_\mu \varepsilon A_\mu^T \varepsilon + A_\mu \varepsilon A_\mu^T \varepsilon + A \varepsilon A_{\mu\mu}^T \varepsilon = 0,$$

$$A_{\mu\mu} \varepsilon A^T \varepsilon + 2A_\mu \varepsilon A_\mu^T \varepsilon + A \varepsilon A_{\mu\mu}^T \varepsilon = 0.$$

Since $rank\ (A_{\mu\mu}) = 0$, we get $A_{\mu\mu} = 0$ and $A_{\mu\mu}^T = 0$, thus we have

$$A_\mu \varepsilon A_\mu^T \varepsilon = 0 \Rightarrow A_\mu \varepsilon A_\mu^T = 0.$$

Since it is true for every $x \in IR_n^1$, we have the following

$$\left(A_\mu \varepsilon A_\mu^T\right) x^T = (0)\, x^T$$

$$\left(A_\mu \varepsilon A_\mu^T\right) x^T = 0$$

$$x \left(A_\mu \varepsilon A_\mu^T\right) x^T = 0$$

$$(x A_{\mu\lambda})\, \varepsilon\, (x A_\mu)^T = 0.$$

Hence

$$\langle x A_\mu, x A_\mu \rangle_L = 0$$

$$x A_\mu = 0.$$

For every $x \in IR_n^1$ it is true, we get $A_\mu = 0$ and $rank\,(A_\mu) = 0$. Conversely it is obviously to see.

§3. Special case $n = 3$

Since A_λ and A_μ are semi skew-symmetric matrices especially

$$A_\lambda = \begin{bmatrix} 0 & j_3 & -j_2 \\ j_3 & 0 & -j_1 \\ -j_2 & j_1 & 0 \end{bmatrix}, A_\mu = \begin{bmatrix} 0 & i_3 & -i_2 \\ i_3 & 0 & -i_1 \\ -i_2 & i_1 & 0 \end{bmatrix}.$$

Let $A_\lambda = -\varepsilon A_\lambda^T \varepsilon, A_\mu = -\varepsilon A_\mu^T \varepsilon$. The equation $X = A^{-1}(\lambda,\mu) Y(\lambda,\mu)$ is obtained from the equation of $Y(\lambda,\mu) = A(\lambda,\mu)X$. By differentiating the equation $Y(\lambda,\mu) = A(\lambda,\mu)X$ with respect to t , we have

$$\begin{aligned} Y_\lambda \dot{\lambda} + Y_\mu \dot{\mu} &= \left(A_\lambda \dot{\lambda} + A_\mu \dot{\mu}\right) X \\ &= \left(A_\lambda \dot{\lambda} + A_\mu \dot{\mu}\right) A^{-1}(\lambda,\mu) Y(\lambda,\mu). \end{aligned}$$

In the position $(\lambda,\mu) = (0,0)$, we have

$$\begin{aligned} Y_\lambda \dot{\lambda} + Y_\mu \dot{\mu} &= \left(A_\lambda \dot{\lambda} + A_\mu \dot{\mu}\right) Y(\lambda,\mu) \\ &= \Omega Y(\lambda,\mu). \end{aligned}$$

Since A_λ and A_μ are semi skew-symmetric matrices, we get

$$\Omega = \begin{bmatrix} 0 & \left(j_3 \dot{\lambda} + i_3 \dot{\mu}\right) & -\left(j_2 \dot{\lambda} + i_2 \dot{\mu}\right) \\ \left(j_3 \dot{\lambda} + i_3 \dot{\mu}\right) & 0 & -\left(j_1 \dot{\lambda} + i_1 \dot{\mu}\right) \\ -\left(j_2 \dot{\lambda} + i_2 \dot{\mu}\right) & \left(j_1 \dot{\lambda} + i_1 \dot{\mu}\right) & 0 \end{bmatrix}$$

and also angular velocity matrix is an semi skew-symmetric. Since A matrix is semi orthogonal, we have the following equation

$$A(\lambda,\mu)\varepsilon A^T(\lambda,\mu)\varepsilon = I.$$

Now differentiating with respect to t, it follows that

$$\left(A_\lambda\dot{\lambda} + A_\mu\dot{\mu}\right)\varepsilon A^T(\lambda,\mu)\varepsilon + A(\lambda,\mu)\varepsilon\left(A_\lambda^T\dot{\lambda} + A_\mu^T\dot{\mu}\right)\varepsilon = 0.$$

For $(\lambda,\mu) = (0,0)$, we obtain

$$\left(A_\lambda\dot{\lambda} + A_\mu\dot{\mu}\right) + \varepsilon\left(A_\lambda\dot{\lambda} + A_\mu\dot{\mu}\right)^T\varepsilon = 0.$$

Since $\Omega = A_\lambda\dot{\lambda} + A_\mu\dot{\mu}$, it follows that

$$\Omega + \varepsilon\,\Omega^T\varepsilon = 0,$$

where Ω is semi skew-symmetric matrix. Since pole points which are the points of sliding velocity is zero given by

$$\overrightarrow{V_f} = \left(A_\lambda\dot{\lambda} + A_\mu\dot{\mu}\right)X + \left(C_\lambda\dot{\lambda} + C_\mu\dot{\mu}\right)$$

is pole points of two parameter motion by

$$Y(\lambda,\mu) = A(\lambda,\mu)X + C(\lambda,\mu).$$

The equation

$$\left(A_\lambda\dot{\lambda} + A_\mu\dot{\mu}\right)X + \left(C_\lambda\dot{\lambda} + C_\mu\dot{\mu}\right) = 0 \tag{3.1}$$

can be solution it must be $rank\left(A_\lambda\dot{\lambda} + A_\mu\dot{\mu}\right) = rank\,\Omega = 2$. It follows that

$$\begin{aligned} Y(\lambda,\mu) &= A(\lambda,\mu)\,X + C(\lambda,\mu) \\ A(\lambda,\mu)X &= Y(\lambda,\mu) - C(\lambda,\mu) \\ A^{-1}(\lambda,\mu)A(\lambda,\mu)X &= A^{-1}(\lambda,\mu)\left(Y(\lambda,\mu) - C(\lambda,\mu)\right) \\ X &= A^{-1}(\lambda,\mu)\left(Y(\lambda,\mu) - C(\lambda,\mu)\right). \end{aligned}$$

If we get write this value of X in the equation (3.1), we have

$$\left(A_\lambda\dot{\lambda} + A_\mu\dot{\mu}\right)\left[A^{-1}(\lambda,\mu)\left(Y(\lambda,\mu) - C(\lambda,\mu)\right)\right] + \left(C_\lambda\dot{\lambda} + C_\mu\dot{\mu}\right) = 0. \tag{3.2}$$

In the position (λ,μ) =$(0,0)$, we have

$$\left(A_\lambda\dot{\lambda} + A_\mu\dot{\mu}\right)A^{-1}(\lambda,\mu) = \left(A_\lambda\dot{\lambda} + A_\mu\dot{\mu}\right) = \Omega.$$

And if we say

$$Y(\lambda,\mu) - C(\lambda,\mu) = Y^*(\lambda,\mu),$$

then the equation (3.2) form

$$\Omega \wedge_L Y^*(\lambda,\mu) + \left(C_\lambda\dot{\lambda} + C_\mu\dot{\mu}\right) = 0. \tag{3.3}$$

The solution of equation (3.3) gives fixed pole points of the motion. For the solution of equation (3.3) it must be verified the condition

$$\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \rangle_L = 0.$$

In generally, this condition can't verify. But we can separate two composite velocities $C_\lambda \dot{\lambda} + C_\mu \dot{\mu}$ either orthogonal $\overrightarrow{\Omega}$ or parallel. If W is angular velocity matrix of moving space, then it can find $W = A^{-1}(\lambda, \mu)\Omega$. Since $A^{-1}(0,0) = I$, we have $W = \Omega$ in the position $(\lambda, \mu) = (0,0)$.
Let

$$E = \frac{\Omega}{\|\Omega\|_L}$$

and

$$E^* = \frac{W}{\|W\|_L}.$$

Then we have $E^* = \eta A E$, where $\eta \in IR$. For $(\lambda, \mu) = (0,0)$ and $\eta = 1$, it follows that

$$E^* = E.$$

We can separate two components of the velocity $C_\lambda \dot{\lambda} + C_\mu \dot{\mu}$ which form

$$U = \left(C_\lambda \dot{\lambda} + C_\mu \dot{\mu}\right) - \frac{\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \rangle_L}{\langle \Omega, \Omega \rangle_L} \Omega$$

and

$$V = \frac{\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \rangle_L}{\langle \Omega, \Omega \rangle_L} \Omega,$$

one is orthogonal to vector $\overrightarrow{\Omega}$, another is parallel to vector $\overrightarrow{\Omega}$ respectively, in which

$$C_\lambda \dot{\lambda} + C_\mu \dot{\mu} = \left(C_\lambda \dot{\lambda} + C_\mu \dot{\mu}\right) - \frac{\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \rangle_L}{\langle \Omega, \Omega \rangle_L} \Omega + \frac{\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \rangle_L}{\langle \Omega, \Omega \rangle_L} \Omega.$$

That is

$$\begin{aligned}
\langle \Omega, U \rangle_L &= \left\langle \Omega, \left(C_\lambda \dot{\lambda} + C_\mu \dot{\mu}\right) - \frac{\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \rangle_L}{\langle \Omega, \Omega \rangle_L} \Omega \right\rangle_L \\
&= \left\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \right\rangle_L - \frac{\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \rangle_L}{\langle \Omega, \Omega \rangle_L} \langle \Omega, \Omega \rangle_L \\
&= \left\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \right\rangle_L - \left\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \right\rangle_L \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
V &= \frac{\langle \Omega, C_\lambda \dot{\lambda} + C_\mu \dot{\mu} \rangle_L}{\langle \Omega, \Omega \rangle_L} \\
&= \sigma \overrightarrow{\Omega}.
\end{aligned}$$

If we write velocity U replacing $C_\lambda\dot{\lambda}+C_\mu\dot{\mu}$ in equation (3.3), we have this condition $\langle\Omega,\ U\rangle_L=0$ is verified in this equation. Let $rank\,\Omega=2$,then the system can be solution. Now we get the solution of the equation

$$\Omega\wedge_L\ Y^*(\lambda,\mu)+U=0.$$

Here $\vec{a}\wedge_L(\vec{b}\wedge_L\vec{c})=\langle\overrightarrow{a},\overrightarrow{b}\rangle_L\overrightarrow{c}-\langle\overrightarrow{a},\overrightarrow{c}\rangle_L\overrightarrow{b}$. It follows that

$$\Omega\wedge\ _LY^*(\lambda,\mu)+U=0,$$

$$\Omega\wedge_L(\Omega\wedge_L Y^*(\lambda,\mu)+U)=0,$$

$$\Omega\wedge_L(\Omega\wedge_L Y^*(\lambda,\mu))+\Omega\wedge_L U=0,$$

$$\langle\Omega,\Omega\rangle_L Y^*(\lambda,\mu)-\langle\Omega,Y^*(\lambda,\mu)\rangle_L\Omega+\Omega\wedge_L Y=0,$$

$$Y^*(\lambda,\mu)=\frac{\langle\Omega,Y^*(\lambda,\mu)\rangle_L}{\langle\Omega,\Omega\rangle_L}\Omega-\frac{\Omega\wedge_L U}{\langle\Omega,\Omega\rangle_L},$$

$$Y^*(\lambda,\mu)=-\frac{\Omega\wedge_L U}{\langle\Omega,\Omega\rangle_L}+\sigma\,\Omega.$$

If we write the last equation replacing by the equation

$$Y^*(\lambda,\mu)=Y(\lambda,\mu)-C(\lambda,\mu),$$

so we have

$$Y(\lambda,\mu)=-\frac{\Omega\wedge_L U}{\langle\Omega,\Omega\rangle_L}+C(\lambda,\mu)+\sigma\,\Omega.$$

Hence we get

$$Y(\lambda,\mu)=Q+\sigma\,\Omega,\sigma\in IR. \tag{3.4}$$

This means that it is a line which is passes though point Q and straight Ω. The line is called fixed pole axis in the fixed space, the expression of the fixed pole axis in the moving space find to write instead of value $Y(\lambda,\mu)$ in the equation (1.1), it follows that

$$-\frac{\Omega\wedge_L U}{\langle\Omega,\Omega\rangle_L}+C(\lambda,\mu)+\sigma\,\Omega=A(\lambda,\mu)X+C(\lambda,\mu),$$

$$-\frac{\Omega\wedge_L U}{\langle\Omega,\Omega\rangle_L}+\sigma\,\Omega=A(\lambda,\mu)X,$$

$$X=-A^{-1}(\lambda,\mu)\frac{\Omega\wedge_L U}{\langle\Omega,\Omega\rangle_L}+A^{-1}(\lambda,\mu)\sigma\,\Omega.$$

In the position $(\lambda,\mu)=(0,0)$, it follows that

$$\begin{aligned}X&=-\frac{\Omega\wedge_L U}{\langle\Omega,\Omega\rangle_L}+\sigma\,\Omega,\\ X&=P+\sigma\,\Omega.\end{aligned}$$

If the pole axis of fixed and moving space coincide in the position $(\lambda,\mu)=(0,0)$, we have $P=Q$ and $C(0,0)=0$. Thus lines passes though points Q and P is straight $\overrightarrow{\Omega}$, which are the pole

axis in fixed and moving space. In the position (λ, μ) of motion the lines have a velocity vector with stationary norm, locus of the lines which called instantaneous screw axis. The equations

$$Y = Q + \sigma\,\Omega, \quad \sigma \in IR$$

$$X = P + \sigma\,\Omega, \quad \sigma \in IR$$

depend only on parameters $\dot{\lambda}$ and $\dot{\mu}$. Thus there is ∞^2 the one parameter motion. There are ∞ instantaneous screw axis since the parameters $\dot{\lambda}$ and $\dot{\mu}$ depend only on t [1]. The locus of this screw axis is a ruled surface. Indeed the following equations determine a ruled surface,

$$Y(t,\sigma) = Q(t) + \sigma\,\Omega(t), \quad \sigma \in IR$$

$$X(t,\sigma) = P(t) + \sigma\,\Omega(t), \sigma \in IR.$$

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Cyclic dualizing elements in Girard quantales [1]

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Abstract In this paper, we study the interior structures of Girard quantale and the cyclic dualizing elements of Girard quantale. some equivalent descriptions for Girard quantale are given and an example which shows that the cyclic dualizing element is not unique is given.

Keywords Complete lattice, quantale, Girard quantale, cyclic dualizing element.

§1. Preliminaries

Quantales were introduced by C.J. Mulvey in [1] with the purpose of studying the spectrum of C^*-algebras and the foundations of quantum mechanics. The study of such partially ordered algebraic structures goes back to a series of papers by Ward and Dilworth [2, 3] in the 1930s. It has become a useful tool in studying noncommutative topology, linear logic and C^*-algebra theory [4-6]. Following Mulvey, various types and aspects of quantales have been considered by many researchers [7-9]. The importance of quantales for linear logic is revealed in Yetter's work [10]. Yetter has clarified the use of quantales in linear logic and he has introduced the term "Girard quantale". In [11], J. Paseka and D. Kruml have shown that any quantale can be embedded into a unital quantale. In [12], K.I. Rosenthal has proved that every quantale can be embedded into a Girard quantale. Thus, it is important to study Girard quantale. This is the motivation for us to investigate Girard quantale. In the note, we shall study the interior structures of Girard quantale and the cyclic dualizing element in Girard quantales.

We use 1 to denote the top element and 0 the bottom element in a complete lattice. For notions and concepts, but not explained, please to refer to [12].

Definition 1.1. A quantale is a complete lattice Q with an associative binary operation "&" satisfying:

$$a\&(\bigvee b_\alpha) = \bigvee(a\&b_\alpha) \ \text{ and } \ (\bigvee b_\alpha)\&a = \bigvee(b_\alpha\&a)$$

for all $a \in Q, \{b_\alpha\} \subseteq Q$.

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An element $e \in Q$ is called a unit if $a\&e = e\&a = a$ for all $a \in Q$. Q is called unital if Q has the unit e.

Since $a\&\_$ and $\_\&a$ preserve arbitrary sups for all $a \in Q$, they have right adjoints and we shall denote them by $a \longrightarrow_r \_$ and $a \longrightarrow_l \_$ respectively.

Proposition 1.2. Let Q be a quantale, $a, b, c \in Q$. Then

(1) $a\&(a \longrightarrow_r b) \leq b$;

(2) $a \longrightarrow_r (b \longrightarrow_r c) = b\&a \longrightarrow_r c$;

Again, analogous results hold upon replacing $\longrightarrow_r$ by $\longrightarrow_l$.

Definition 1.3. Let Q be a quantale, An element c of Q is called cyclic, if $a \longrightarrow_r c = a \longrightarrow_l c$ for all $a \in Q$. $d \in Q$ is called a dualizing element, if $a = (a \longrightarrow_l d) \longrightarrow_r d = (a \longrightarrow_r d) \longrightarrow_l d$ for all $a \in Q$.

Definition 1.4. A quantale Q is called a Girard quantale if it has a cyclic dualizing element d.

Let Q be a Girard quantale with cyclic dualizing element d and $a, b \in Q$, define the binary operation "$\|$ " by $a\|b = (a^{\perp}\&b^{\perp})^{\perp}$, then we can prove that $a\|\_$ and $\_\|a$ preserve arbitrary infs for all $a \in Q$, hence they have left adjoints and we shall denote them by $a \longmapsto_r \_$ and $a \longmapsto_l \_$ respectively. If $a \longrightarrow_r d = a \longrightarrow_l d$, we shall denote it by $a \longrightarrow d$, or more frequently by $a^{\perp}$ if d is a cyclic dualizing element.

§2. The equivalent descriptions for Girard quantale

In this section, we shall study the interior structures of Girard quantale and give some equivalent descriptions for Girard quantale. According to the above, we know that there are six binary operations on a Girard quantale such as $\_\&\_$, $\_\longrightarrow_r \_$, $\_\longrightarrow_l \_$, $\_\|\_$, $\_\longmapsto_r \_$, $\_\longmapsto_l \_$, we shall respectively call them multiplying, right implication, left implication, Par operation, dual right implication and dual left implication for convenience.

Theorem 2.1. Let Q be a unital quantale, $^{\perp} : Q \longrightarrow Q$ an unary operation on Q. Then Q is a Girard quantale if and only if

(1) $a \longrightarrow_l b = (a\&b^{\perp})^{\perp}$; (2) $a \longrightarrow_r b = (b^{\perp}\&a)^{\perp}$.

Proof. The necessity is obvious. Sufficiency: suppose (1) and (2) hold, $a \in Q$. Denote the unit element by e on Q, then $a = e \longrightarrow_l a = (e\&a^{\perp})^{\perp} = (a^{\perp})^{\perp}$. Thus $a = a^{\perp\perp}$, hence

$$a \longrightarrow_l e^{\perp} = (a\&e^{\perp\perp})^{\perp} = (a\&e)^{\perp} = a^{\perp}.$$

Similarly we get $a^{\perp} = a \longrightarrow_r e^{\perp}$. Take $d = e^{\perp}$, thus d is a cyclic element of Q. Again $\forall a \in Q, (a \longrightarrow e^{\perp}) \longrightarrow e^{\perp} = a^{\perp} \longrightarrow e^{\perp} = (a^{\perp})^{\perp} = a$. This proves $d = e^{\perp}$ is a dualizing element on Q. Thus the proof is completed.

Theorem 2.2. Let Q be a complete lattice. $\_\longrightarrow_r \_ : Q \times Q \longrightarrow Q$ is a binary operation on Q, $a \longrightarrow_r \_ : Q \longrightarrow Q$ and $\_ \longrightarrow_r a : Q \longrightarrow Q^{op}$ preserve arbitrary sups for all $a \in Q$. $^{\perp} : Q \longrightarrow Q$ is a unary operation on Q, $e \in Q$. For all $a, b, c \in Q$,

(1) $e \longrightarrow_r a = a$; $a \longrightarrow_r e^{\perp} = a^{\perp}$;

(2) $(a^{\perp})^{\perp} = a$; $a \leq b \Longrightarrow b^{\perp} \leq a^{\perp}$;

(3) $(a \longrightarrow_r b^{\perp})^{\perp} \longrightarrow_r c = a \longrightarrow_r (b \longrightarrow_r c)$;

(4) $c \le a \longrightarrow_r b^{\perp} \Longleftrightarrow a \le b \longrightarrow_r c^{\perp}$.

Then Q is a Girard quantale and $\_ \longrightarrow_r \_$ is the right implication operation.

Proof. Define the binary operation $a\&b = (b \longrightarrow_r a^{\perp})^{\perp}$ for all $a, b \in Q$.

& satisfies associative law: In fact, for all $a, b, c \in Q$,

$$(a\&b)\&c = (b \longrightarrow_r a^{\perp})^{\perp}\&c = (c \longrightarrow_r (b \longrightarrow_r a^{\perp}))^{\perp}.$$

Using the condition (3), we have $(c \longrightarrow_r (b \longrightarrow_r a^{\perp}))^{\perp} = ((c \longrightarrow_r b^{\perp})^{\perp} \longrightarrow_r a^{\perp})^{\perp}$. By the definition of the binary operation & we can get

$$a\&(b\&c) = a\&(c \longrightarrow_r b^{\perp})^{\perp} = ((c \longrightarrow_r b^{\perp})^{\perp} \longrightarrow_r a^{\perp})^{\perp}.$$

So $(a\&b)\&c = a\&(b\&c)$.

Using the condition (4), we have $a\&b \le c \Longleftrightarrow (b \longrightarrow_r a^{\perp})^{\perp} \le c \Longleftrightarrow c^{\perp} \le b \longrightarrow_r a^{\perp} \Longleftrightarrow b \le a \longrightarrow_r c$ for all $a, b, c \in Q$.

For any $a \in Q, \{b_i\}_{i \in I} \subseteq Q$. If $I = \emptyset$, then $a\&0 = (a \longrightarrow_r 0^{\perp})^{\perp} = (a \longrightarrow_r 1)^{\perp}$, since again $(a \longrightarrow_r 1)^{\perp} \le 0 \Longleftrightarrow 1 \le a \longrightarrow_r 1 \Longleftrightarrow a\&1 \le 1$, the last inequality obviously holds. So $a\&0 = 0$. Thus $a\&\_$ preserves empty-sups. If $I \ne \emptyset$, then

$$\begin{aligned} a\&(\bigvee_{i \in I} b_i) &= ((\bigvee_{i \in I} b_i) \longrightarrow_r a^{\perp})^{\perp} \\ &= (\bigwedge_{i \in I}(b_i \longrightarrow_r a^{\perp}))^{\perp} \\ &= \bigvee_{i \in I}(b_i \longrightarrow_r a^{\perp})^{\perp} \\ &= \bigvee_{i \in I}(a\&b_i). \end{aligned}$$

Hence $a\&\_$ preserves arbitrary sups for all $a \in Q$. Similarly, we can prove $\_\&a$ preserves arbitrary sups for all $a \in Q$. Thus $(Q, \&)$ is a quantale.

In accordance with the condition (1), we know e is the unit element corresponding to & on Q and $a \longrightarrow_r e^{\perp} = a^{\perp}$. Denote by $a \longrightarrow_l \_$ the right adjoint of $\_\&a$. Then

$$\begin{aligned} a \longrightarrow_l e^{\perp} &= \bigvee\{x \in Q | x \le a \longrightarrow_l e^{\perp}\} \\ &= \bigvee\{x \in Q | x\&a \le e^{\perp}\} \\ &= \bigvee\{x \in Q | (a \longrightarrow_r x^{\perp})^{\perp} \le e^{\perp}\} \\ &= \bigvee\{x \in Q | e \le a \longrightarrow_r x^{\perp}\} \\ &= \bigvee\{x \in Q | a\&e \le x^{\perp}\} \\ &= \bigvee\{x \in Q | a \le x^{\perp}\} \\ &= \bigvee\{x \in Q | x \le a^{\perp}\} \\ &= a^{\perp}. \end{aligned}$$

This show $e^{\perp}$ is a cyclic element in Q. Using conditions (1) and (2) we know $e^{\perp}$ is also a dualizing element on Q. Hence $(Q, \&, \perp)$ is a Girard quantale. We can easily prove $\_ \longrightarrow_r \_$ is the right implication operation on Q by the above consideration.

Theorem 2.3. Let Q be a complete lattice. $\_ \longmapsto_r \_ : Q \times Q \longrightarrow Q$ is a binary operation in Q, $a \longmapsto_r \_ : Q \longrightarrow Q$ and $\_ \longmapsto_r a : Q^{op} \longrightarrow Q$ preserve arbitrary sups for all $a \in Q$. $^{\perp} : Q \longrightarrow Q$ is an unary operation in Q, $d \in Q$. For all $a, b, c \in Q$,

(1) $d \longmapsto_r a = a$; $a \longmapsto_r d^\perp = a^\perp$;
(2) $(a^\perp)^\perp = a$; $a \leq b \Longrightarrow b^\perp \leq a^\perp$;
(3) $(a \longmapsto_r b)^\perp \longmapsto_r c = a \longmapsto_r (b^\perp \longmapsto_r c)$;
(4) $c \geq a \longmapsto_r b \Longleftrightarrow b^\perp \geq c \longmapsto_r a^\perp$.

Then Q is a Girard quantale and $\_ \longmapsto_r \_$ is the dual right implication operation.

Proof. Define binary operation $a\&b = (b^\perp \longmapsto_r a)$ for all $a, b \in Q$,

(i) The binary operation & is associative: Since $\forall a, b, c \in Q$,

$$\begin{aligned}(a\&b)\&c &= (b^\perp \longmapsto_r a)\&c\\ &= c^\perp \longmapsto_r (b^\perp \longmapsto_r a)\\ &= (c^\perp \longmapsto_r b)^\perp \longmapsto_r a\\ &= a\&(c^\perp \longmapsto_r b)\\ &= a\&(b\&c).\end{aligned}$$

(ii) Using the condition (2), we can prove

$$(\bigvee_{i\in I} a_i)^\perp = \bigwedge_{i\in I}(a_i)^\perp; \quad (\bigwedge_{i\in I} a_i)^\perp = \bigvee_{i\in I}(a_i)^\perp$$

for any set I and $\{a_i\}_{i\in I} \subseteq Q$.

(iii) For all $a \in Q, \{b_i\}_{i\in I} \subseteq Q$, we have

$$a\&(\bigvee_{i\in I} b_i) = (\bigvee_{i\in I} b_i)^\perp \longmapsto_r a = \bigwedge_{i\in I}(b_i)^\perp \longmapsto_r a = \bigvee_{i\in I}(b_i^\perp \longmapsto_r a) = \bigvee_{i\in I}(a\&b_i).$$

Similarly we have $(\bigvee_{i\in I} b_i)\&a = \bigvee_{i\in I}(b_i\&a)$. Hence $(Q,\&)$is a quantale. Since $a\&d^\perp = (d^\perp)^\perp \longmapsto_r a = d \longmapsto_r a = a$; $d^\perp\&b = b^\perp \longmapsto_r d^\perp = b$, thus $(Q,\&)$ is a unit quantale with unit element $d^\perp$.

(iv) If $a \in Q$, we have

$$\begin{aligned}a \longrightarrow_l d &= \bigvee\{x \in Q | x \leq a \longrightarrow_l d\}\\ &= \bigvee\{x \in Q | x\&a \leq d\}\\ &= \bigvee\{x \in Q | a^\perp \longmapsto_r x \leq d\}\\ &= \bigvee\{x \in Q | d \longmapsto_r a \leq x^\perp\}\\ &= \bigvee\{x \in Q | x \leq a^\perp\}\\ &= a^\perp.\end{aligned}$$

Similarly, $a \longrightarrow_r d = a^\perp$, hence d is a cyclic element in Q. d is also a dual element in Q by condition (2). Thus (Q, &) is a Girard quantale with cyclic dual element d. We easily know $\_ \longmapsto_r \_$ is the dual right implication operation in Q by the definition of &.

Obviously, Theorem 2.2 and Theorem 2.3 also hold if $\longrightarrow_r$ and $\longmapsto_r$ are substituted by $\longrightarrow_l$ and $\longmapsto_l$ respectively, "right" and "left" replace each other.

Theorem 2.4. Let Q be a unital quantale with a unary operation $^\perp$ satisfying the condition

$$\textbf{CN}: \quad (a^\perp)^\perp = a \quad \text{and} \quad a \longrightarrow_r b = b^\perp \longrightarrow_l a^\perp$$

for all $a, b \in Q$. Then Q is a Girard quantale.

§3. The cyclic dualizing element of Girard quantale

According to the definition of Girard quantale, we know that the cyclic dualizing element plays an important role in Girard quantale, so we shall discuss the cyclic dualizing element in this section. We shall account for whether the cyclic dualizing element is unique in a Girard quantale; when it is unique; whether these Girard quantales determined by different cyclic dualizing elements are different. Let us see the following example

Example 3.1. Let $Q = \{0, a, b, c, 1\}$, the partial order on Q be defined as Fig 1, the operator & on Q be defined by Table 1. Then we can prove that Q is a commutative Girard quantale. And we can prove that a, b and c are cyclic dualizing elements of Q.

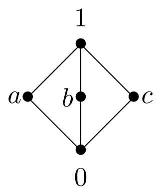

Fig 1

&	0	a	b	c	1
0	0	0	0	0	0
a	0	b	c	a	1
b	0	c	a	b	1
c	0	a	b	c	1
1	0	1	1	1	1

Table 1

Proposition 3.2. Let Q be a unital quantale with the unit element e. $^{\perp_1}$ and $^{\perp_2}$ satisfy the condition **CN** in Theorem 2.4. Then $e^{\perp_1} = e^{\perp_2}$ if and only if $^{\perp_1} = ^{\perp_2}$.

Proposition 3.3. Let Q be a quantale, d_1, d_2 are cyclic dualizing elements of Q, $^{\perp_1}, ^{\perp_2}$ are unary operations on Q induced by d_1, d_2 respectively. Then $d_1 = d_2$ if and only if $^{\perp_1} = ^{\perp_2}$.

Theorem 3.4. Let Q be a Girard quantale. Then there is a one-to-one correspondence between the set of cyclic dualizing elements in Q and the set of unary operations satisfying the condition **CN** in Theorem 2.4.

Proposition 3.5. Let Q be a Girard quantale. If 0 is a cyclic dualizing element of Q, then Q is strictly two-sided.

Proof. Assume 0 is a cyclic dualizing element in Q. Then $0^{\perp} = 0 \longrightarrow 0 = 1$ is the unit of Q, hence $\forall a \in Q, a\&1 = 1\&a = a$, this finished the proof.

Proposition 3.6. If Q is a two-sided Girard quantale, then the unique cyclic dualizing element is the least element 0.

Proof. If Q is a two-sided Girard quantale, then we have $a = a\&e \leq a\&1 \leq a$ for all $a \in Q$. Similarly, we have $1\&a = a$. Thus Q is strictly two-sided. Suppose d is a cyclic dual element in Q, $^{\perp}$ is the unary operation induced by d, then we have $d = 1 \longrightarrow d = 1^{\perp} = 0$. the proof is finished.

Corollary 3.7. Let Q be a Girard quantale with cyclic dualizing element 0. Then the cyclic dualizing element of Q is unique.

Theorem 3.8. Any complete lattice implication algebra is a Girard quantale with unique cyclic dualizing element 0.

According the above conclusions, we have a question : Whether the cyclic dualizing element must be the least element 0 if a Girard quantale has an unique cyclic dualizing element. The

answer is negative. Let us see the following example.

Example 3.9. Let $Q = \{0, e, 1\}$, the partial order on Q be defined by $0 < e < 1$, the binary operation & be defined by Table 2

&	0	e	1
0	0	0	0
e	0	e	1
1	0	1	1

Table 2

It is immediate to verify Q being a Girard quantale with the unique cyclic dualizing element e.

Question 3.10. What is the necessary condition when the cyclic dualizing element of Girard quantale is unique?

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$\nu-$Compact spaces

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Abstract In this paper $\nu-$compactness and $\nu-$Lindeloffness in topological space are introduced, obtained some of its basic properties and interrelations are verified with other types of compactness and Lindeloffness.

Keywords $\nu-$compact, $\nu-$Lindeloff spaces

§1. Introduction

After the introduction of semi open sets by Norman Levine various authors have turned their attentions to this concept and it becomes the primary aim of many mathematicians to examine and explore how far the basic concepts and theorems remain true if one replaces open set by semi open set. The concept of semi compactness was introduced by C. Dorsett in 1980. After him Reilly and Vamanamurthy studied about semi compactness during 1984. U.N. B. Dissanayake and K. P. R. Sastry introduced locally Lindeloff spaces. In the present paper we introduce the concepts of compactness and lindeloffness using $\nu-$open sets in topological spaces.

Throughout the paper a space X means a topological space (X, τ). The class of $\nu-$open sets is denoted by $\nu - O(X, \tau)$ respectively. The interior, closure, $\nu-$interior, $\nu-$closure are defined by A^o, A^-, νA^o, νA^- .

In section 2 we discuss the basic definitions and results used in this paper. In sections 3 and 4 we discuss about the $\nu-$compact and $\nu-$Lindeloffness in the topological space and obtain their basic properties.

§2. Preliminaries

A subset A of a topological space (X, τ) is said to be regularly open if $A = ((A)^-)^o$, semi open(regularly semi open or $\nu-$open) if there exists an open(regularly open) set O such that $O \subset A \subset (O)^-$ and $\nu-$closed if its complement is $\nu-$open. The intersection of all $\nu-$closed sets containing A is called $\nu-$closure of A, denoted by $\nu(A)^-$. The class of all $\nu-$closed sets are denoted by $\nu - CL(X, \tau)$. The union of all $\nu-$open sets contained in A is called the $\nu-$interior

of A,denoted by $\nu(A)^o$. A function f: (X, τ) $\rightarrow$ (Y, σ) is said to be $\nu-$continuous if the inverse image of any open[closed]set in Y is a $\nu-$open[$\nu-$closed]set in X. $\nu-$irresolute if the inverse image of any $\nu-$open[$\nu-$closed]set in Y is a $\nu-$open[$\nu-$closed] set in X. $\nu-$open[$\nu-$closed] if the image of every $\nu-$open[$\nu-$closed]set is $\nu-$open[$\nu-$closed]. $\nu-$homeomorphism if f is bijective, $\nu-$irresolute and $\nu-$open. Let x be a point of (X, τ) and V be a subset of X, then V is said to be $\nu-$neighbourhood of x if there exists a $\nu-$open set U of X such that $x \in U \subset V$. $x \in X$ is said to be $\nu-$limit point of U iff for each $\nu-$open set V containing x $V \cap (U - \{x\}) \neq \phi$. The set of all $\nu-$limit points of U is called $\nu-$derived set of U and is denoted by $D_\nu(U)$.

Note 1. Clearly every regularly open set is $\nu-$open and every $\nu-$open set is semi-open but the reverse implications do not holds good. that is, RO(X)$\subset \nu - O(X) \subset$SO(X).

Theorem 2.1. If x is a $\nu-$limit point of any subset A of the topological space (X, τ), then every $\nu-$neighbourhood of x contains infinitely many distinct points.

Theorem 2.2. (i) union and intersection of any two $\nu-$open sets is not $\nu-$open.

(ii) Intersection of a regular open set and a $\nu-$open set is $\nu-$open.

(iii) If $B \subset$X such that $A \subset B \subset (A)^-$ then B is $\nu-$open iff A is $\nu-$open.

(iv) If A and R are regularly open and S is $\nu-$open such that $R \subset S \subset (R)^-$. Then $A \cap$R $= \phi \Rightarrow A \cap S = \phi$.

Theorem 2.3. In a semi regular space, int $\nu - O(X, \tau)$ generates topology.

Theorem 2.4. (i) Let $A \subseteq Y \subseteq X$ and Y is regularly open subspace of X then A is $\nu-$open in X iff A is $\nu-$open in $\tau_{/Y}$.

(ii) Let $Y \subseteq X$ and $A \in \nu - O(Y, \tau_{/Y})$ then A$\in \nu-$O(X,τ) iff Y is $\nu-$open in X.

(iii) Let $Y \subseteq X$ and $A$is a $\nu-$neighborhood of x in . Then A is a $\nu-$neighborhood of x in Y iff Y is $\nu-$open in X.

Theorem 2.5. An almost continuous and almost open map is $\nu-$irresolute.

Example 1. Identity map is $\nu-$irresolute.

Remark 1. For any topological space we have the following interrelations.

(i) compact $\Rightarrow$ nearly-compact $\Rightarrow$ almost compact $\Rightarrow$ weakly compact.

(ii)compact $\Rightarrow$ semi-compact where none of the implications is reversible.

§3. $\nu-$Compact spaces

Definition 3.1. A space X is said to be

(i) $\nu-$compact space if every $\nu-$open cover of it has a finite sub cover.

(ii) Countably $\nu-$compact space if every countable $\nu-$open cover of it has a finite sub cover.

(iii) $\sigma - \nu-$compact if it is the countable union of $\nu-$compact spaces.

Theorem 3.1. Let (X, τ) be a topological space and $A \subseteq X$. Then A is $\nu-$compact subset of X iff the subspace $(A, \tau_{/A})$ is $\nu-$compact.

Theorem 3.2. (i) $\nu-$closed subset of a (countably) $\nu-$compact space is (countably) $\nu-$compact.

(ii) A $\nu-$irresolute image of a (countably) $\nu-$compact space is (countably) $\nu-$compact.

(iii) countable product of (countably) $\nu-$compact spaces is (countably) $\nu-$compact.

(iv) countable union of (countably) $\nu-$compact spaces is (countably) $\nu-$compact.

Remark 2. (countably) $\nu-$compactness is a weakly hereditary property.

Theorem 3.3. For a $\nu - T_1$ topological space X the following statements are equivalent

(i) X is countably $\nu-$compact.

(ii)Every countable family of $\nu-$closed subsets of X which has the finite intersection property has a non-empty intersection.

(iii)Every infinite subset has an $\nu-$accumulation point.

(iv) Every sequence in X has a $\nu-$limit point.

(v) Every infinite $\nu-$open cover has a proper sub cover

Theorem 3.4. Every $\nu-$irresolute map from a $\nu-$compact space into a $\nu - T_2-$space is $\nu-$closed.

Proof. Suppose $f: X \to Y$ is $\nu-$irresolute where X is $\nu-$compact and Y is $\nu - T_2$. Let C be any $\nu-$closed subset of X. Then C is $\nu-$compact and so f(C) is $\nu-$compact. But then f(C) is $\nu-$closed in Y (by Theorem 3.2). Hence the image of any $\nu-$closed set in X is $\nu-$closed set in Y. Thus f is $\nu-$closed.

Theorem 3.5. An $\nu-$continuous bijection from a $\nu-$compact space onto a $\nu - T_2-$space is a $\nu-$homeomorphism.

Proof. Let $f: X \to Y$ be a $\nu-$continuous bijection from a $\nu-$compact space onto a $\nu - T_2$-space. Let G be an $\nu-$open subset of X. Then X-G is $\nu-$closed and hence f(X-G) is $\nu-$closed (by Th 3.2). Since f is bijective $f(X - G) = Y - f(G)$. Therefore f(G) is $\nu-$open in Y implies f is $\nu-$open. Hence f is bijective $\nu-$irresolute and $\nu-$open. Thus f is $\nu-$homeomorphism.

Definition 3.2. A space X is said to be Locally $\nu-$compact space if every $x \in X$ has a $\nu-$ neighborhood whose closure is νcompact.

Note 2. Every $\nu-$compact space is locally $\nu-$compact.

Theorem 3.6. If $f: (X, \tau) \to (Y, \sigma)$ is $\nu-$irresolute, $\nu-$open and X is locally $\nu-$ compact, then so is Y.

Proof. Let $y \in Y$. Then $\exists x \in X \ni f(x) = y$. Since X is locally $\nu-$compact x has a $\nu-$compact neighborhood V Then by $\nu-$irresolute, $\nu-$open of $f, f(V)$ is a $\nu-$compact neighborhood of y. Hence Y is $\nu-$ compact.

Corollary 1. If $f: (X, \tau) \to (Y, \sigma)$ is $\nu-$irresolute, $\nu-$open and X is $\nu-$ compact, then Y is Locally $\nu-$compact.

Proof. Obvious from above two theorems.

Theorem 3.7. Let (X, τ) be a topological space and $A \subseteq X$. Then A is locally $\nu-$compact subset of X iff the subspace $(A, \tau_{/A})$ is locally $\nu-$compact.

Theorem 3.8. (i) $\nu-$closed subset of a locally $\nu-$Compact space is locally $\nu-$Compact.

(ii) countable product of locally $\nu-$Compact spaces is locally $\nu-$Compact.

(iii) countable union of locally $\nu-$Compact spaces is locally $\nu-$Compact.

Theorem 3.9. For a topological space X, the following are equivalent

(i) X is $\nu-$compact.

(ii) Every family of $\nu-$closed subsets of X, having empty intersection has a finite subclass with empty intersection.

(iii) Every family of $\nu-$closed subsets of X which has the finite intersection property(f.i.p) has a non-empty intersection.

Theorem 3.10. Every $\nu-$compact, $\nu-$Hausdorff space is almost $\nu-$regular.

Theorem 3.11. Every pair of disjoint $\nu-$compact subsets of a Hausdorff space have disjoint $\nu-$open neighbourhoods.

From the definitions and remark 1, we have the following:

Remark 3. For any topological space
nearly-compact $\Rightarrow$ $\nu-$compact $\Rightarrow$ semi-compact but the converse is not true in general.

Weakening covering condition from finite cover to countable cover, we have the following as a easy consequence of the above section.

§4. $\nu-$Lindeloff and locally $\nu-$Lindeloff spaces

In this section we define Lindeloffness using $\nu-$open sets their properties and characterizations are verified

Definition 4.1. A space (X, τ) is said to be

(i) $\nu-$Lindeloff space if every $\nu-$open cover of it has a countable sub cover.

(ii) $\sigma-\nu-$Lindeloff if it is the countable union of $\nu-$Lindeloff spaces.

Theorem 4.1. Let (X, τ) be a topological space and $A \subseteq X$. Then A is $\nu-$Lindeloff subset of X iff the subspace $(A, \tau_{/A})$ is $\nu-$Lindeloff.

Theorem 4.2. (i) $\nu-$closed subset of a $\nu-$Lindeloff space is $\nu-$Lindeloff.

(ii) countable product of $\nu-$Lindeloff spaces is $\nu-$Lindeloff.

(iii) countable union of $\nu-$Lindeloff spaces is $\nu-$Lindeloff.

Definition 4.2. A space (X, τ) is said to be locally $\nu-$Lindeloff space if every $x \in X$ has a $\nu-$Lindeloff neighborhood.

Note 3. Every $\nu-$Lindeloff space is locally $\nu-$Lindeloff.

Theorem 4.3. If $f\colon (X, \tau) \to (Y, \sigma)$ is $\nu-$irresolute, $\nu-$open and X is locally $\nu-$Lindeloff, then so is Y.

Proof. Let $y \in Y$. Then $\exists x \in X \ni f(\mathrm{x}) = \mathrm{y}$. Since X is locally $\nu-$Lindeloff x has a $\nu-$Lindeloff neighborhood V Then by $\nu-$irresolute, $\nu-$ open of *f*, *f*(V) is a $\nu-$Lindeloff neighborhood of y. Hence Y is $\nu-$Lindeloff.

Corollary 2. If $f\colon (X, \tau) \to (Y, \sigma)$ is $\nu-$irresolute, $\nu-$open and X is $\nu-$Lindeloff, then Y is Locally $\nu-$Lindeloff.

Theorem 4.4. Let (X, τ) be a topological space and $A \subseteq X$. Then A is locally $\nu-$Lindeloff subset of X iff the subspace $(A, \tau_{/A})$ is locally $\nu-$Lindeloff.

Theorem 4.5. (i) $\nu-$closed subset of a locally $\nu-$Lindeloff space is locally $\nu-$Lindeloff

(ii) countable product of locally $\nu-$Lindeloff spaces is locally $\nu-$Lindeloff.

(iii) countable union of locally $\nu-$Lindeloff spaces is locally $\nu-$Lindeloff.

Theorem 4.6. For a Topological space X, the following are equivalent

(i) X is $\nu-$Lindeloff.

(ii) Every family of $\nu-$ closed subsets of X, having empty intersection has a countable subclass with empty intersection.

(iii) Every family of $\nu-$closed subsets of X which has the countable intersection condition(c.i.c) has a non-empty intersection.

Remark 4. For any topological space we have the following interrelations.

(i) Lindeloff $\Rightarrow$ nearly-Lindeloff $\Rightarrow$ almost Lindeloff $\Rightarrow$ weakly Lindeloff.

(ii)Lindeloff $\Rightarrow$ semi-Lindeloff where none of the implications is reversible.

Remark 5. For any topological space
nearly-Lindeloff $\Rightarrow$ $\nu-$Lindeloff $\Rightarrow$ semi-Lindeloff but the converse is not true in general.

Remark 6.

$$\begin{array}{ccc} \nu-\text{compact} \Rightarrow & & \nu-\text{Lindeloff} \\ \Downarrow & & \Downarrow \\ \text{locally } \nu-\text{compact} \Rightarrow & & \text{locally } \nu-\text{Lindeloff.} \end{array}$$

none is reversible

Conclusion.

In this paper we defined new compact and lindeloff axioms using $\nu-$open sets and studied their interrelations with other compact and lindeloff axioms.

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$\vee$ and $\wedge-$sets of generalized Topologies

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Abstract We define $\wedge_\kappa-$sets, $\vee_\kappa-$sets, $g.\wedge_\kappa$ $-$sets and $g.\vee_\kappa$ $-$sets and characterize these sets for the collection $\Omega = \{\mu,\ \alpha,\ \sigma,\ \pi,\ b,\ \beta\}$ of generalized topologies. For each $\kappa \in \Omega$, we also define and characterize the separation axioms $\kappa - T_i, i = 0, 1, 2$ and $\kappa - R_i, i = 0, 1$.

Keywords $\gamma-$open, $\gamma-$closed, $\gamma-$semiopen, $\gamma-$preopen, $\gamma\alpha-$open, $\gamma\beta-$open and $\gamma b-$open sets, generalized topology.

§1. Introduction

In 1997, Professor *Á. Császár* [1] nicely presented the open sets and all weak forms of open sets in a topological space X in terms of monotonic functions defined on $\wp(X)$, the collection of all subsets of X. For each such function γ, he defined a collection μ of subsets of X, called the collection of $\gamma-$*open* sets. A is said to be $\gamma-$*open* if $A \subset \gamma(A)$. B is said to be $\gamma-$*closed* if its complement is $\gamma-$open. With respect to this collection μ of subsets of X, for $A \subset X$, the $\gamma-$interior of A, denoted by $i_\gamma(A)$, is defined as the largest $\gamma-$open set contained in A and the $\gamma-$closure of A, denoted by $c_\gamma(A)$, is the smallest $\gamma-$closed set containing A. It is established that μ is a generalized topology [3]. In [5], $\gamma-$semiopen sets are defined and discussed. In [7], $\gamma\alpha-$open sets, $\gamma-$preopen sets and $\gamma\beta-$open sets are defined and discussed. $\gamma b-$open sets are defined in [6]. If α is the family of $\gamma\alpha-$open sets, σ is the family of all $\gamma-$semiopen sets, π is the family of all $\gamma-$preopen sets, b is the family of all $\gamma b-$open sets and β is the family of all $\gamma\beta-$open sets, then each collection is a generalized topology. Since every topological space is a generalized topological space, we prove that some of the results established for topological spaces are also true for the generalized topologies $\Omega = \{\mu,\ \alpha,\ \sigma,\ \pi,\ b,\ \beta\}$. In section 2, we list all the required definitions and results. In section 3, we define the $\wedge_\kappa$ and $\vee_\kappa$ operators for each $\kappa \in \Omega$ and discuss its properties. Then, we define $\wedge_\kappa-$sets, $\vee_\kappa-$sets, $g.\wedge_\kappa$ $-$sets and $g.\vee_\kappa$ $-$sets and characterize these sets. In section 4, for each $\kappa \in \Omega$, we define and characterize the separation axioms $\kappa - T_i, i = 0, 1, 2$ and $\kappa - R_i, i = 0, 1$.

§2. Preliminaries

Let X be a nonempty set and $\Gamma = \{\gamma : \wp(X) \to \wp(X) \mid \gamma(A) \subset \gamma(B)$ whenever $A \subset B\}$. For $\gamma \in \Gamma$, a subset $A \subset X$ is said to be $\gamma-$*open* [1] if $A \subset \gamma(A)$. The complement of a $\gamma-$open set is said to be a $\gamma-$*closed* set. A family $\xi \subset \wp(X)$ is said to be a *generalized topology* [3] if $\emptyset \in \xi$ and ξ is closed under arbitrary union. The family of all $\gamma-$open sets, denoted by μ, is a generalized topology [4]. $A \subset X$, is said to be $\gamma-$*semiopen* [5] if there is a $\gamma-$open set G such that $G \subset A \subset c_\gamma(G)$ or equivalently, $A \subset c_\gamma i_\gamma(A)$ [8, Theorem 2.4]. A is said to be $\gamma-$*preopen* [7] if $A \subset i_\gamma c_\gamma(A)$. $A \subset X$, is said to be $\gamma\alpha-$*open* [7] if $A \subset i_\gamma c_\gamma i_\gamma(A)$. A is said to be $\gamma\beta-$*open* [7] if $A \subset c_\gamma i_\gamma c_\gamma(A)$. A is said to be $\gamma b-$*open* [6] if $A \subset i_\gamma c_\gamma(A) \cup c_\gamma i_\gamma(A)$. In [4], [5], [6] and [7], it is established that each $\kappa \in \Omega$ is a generalized topology and so c_κ and i_κ can be defined, similar to the the definition of c_γ and i_γ. In this paper, for $\kappa \in \Omega$, the pair (X, κ) is called a generalized topological space or simply a space. For each $\gamma \in \Gamma$, a mapping $\gamma^\star : \wp(X) \to \wp(X)$ [1] is defined by $\gamma^\star(A) = X - \gamma(X - A)$. Clearly, $\gamma^\star \in \Gamma$. The following lemmas will be useful in the sequel. Moreover, one can easily prove the following already established results of Lemma 2.1.

Lemma 2.1. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then the following hold.

(a) $A \subset c_\kappa(A)$ for every subset A of X.

(b) If $A \subset B$, then $c_\kappa(A) \subset c_\kappa(B)$.

(c) For every subset A of X, $x \in c_\kappa(A)$ if and only if there exists a $\kappa-$open set G such that $A \cap G \neq \emptyset$ (For $\kappa = \mu$, it is established in Lemma 2.1 of [2]).

(d) A is $\kappa-$closed if and only if $A = c_\kappa(A)$.

(e) $c_\kappa(A)$ is the intersection of all $\kappa-$closed sets containing A.

Lemma 2.2. Let X be a nonempty set and $\gamma \in \Gamma(X)$. Then X is $\gamma-$semiopen [5, Proposition 1.2].

§3. $\vee-$sets and $\wedge-$sets

In this section, for the space (X, κ), $\kappa \in \Omega$, we define $\vee-$sets, $\wedge-$sets, $g.\vee-$sets and $g.\wedge-$sets. For $\sigma \in \Omega$, $\vee-$sets and $\wedge-$sets are defined in [5]. For $A \subset X$, we define $\wedge_\kappa(A) = \cap\{U \subset X \mid A \subset U$ and $U \in \kappa\}$ and $\vee_\kappa(A) = \cup\{U \subset X \mid U \subset A$ and U is $\kappa-$closed $\}$. The following Theorem 3.1 gives the properties of the operator $\wedge_\kappa$. Example 3.2 below shows that the two sets in 3.1(e) are not equal.

Theorem 3.1. Let A, B and $\{C_\iota \mid \iota \in \Delta\}$ be subsets of X, $\gamma \in \Gamma$ and $\kappa \in \Omega$. Then the following hold.

(a) If $A \subset B$, then $\wedge_\kappa(A) \subset \wedge_\kappa(B)$.

(b)$A \subset \wedge_\kappa(A)$.

(c)$\wedge_\kappa(\wedge_\kappa(A)) = \wedge_\kappa(A)$.

(d)$\wedge_\kappa(\cup\{C_\iota \mid \iota \in \Delta\}) = \cup\{\wedge_\kappa(C_\iota) \mid \iota \in \Delta\}$.

(e)$\wedge_\kappa(\cap\{C_\iota \mid \iota \in \Delta\}) \subset \cap\{\wedge_\kappa(C_\iota) \mid \iota \in \Delta\}$.

(f) If $A \in \kappa$, then $\wedge_\kappa(A) = A$.

(g)$\wedge_\kappa(A) = \{x \mid c_\kappa(\{x\}) \cap A \neq \emptyset\}$.

(h)$y \in \wedge_\kappa(\{x\})$ if and only if $x \in c_\kappa(\{y\})$.

(i)$\wedge_\kappa(\{x\}) \neq \wedge_\kappa(\{y\})$ if and only if $c_\kappa(\{x\}) \neq c_\kappa(\{y\})$.

Proof.

(a). Suppose $x \notin \wedge_\kappa(B)$. Then there exists $G \in \kappa$ such that $B \subset G$ and $x \notin G$. Since $A \subset B$, there exists $G \in \kappa$ such that $A \subset G$ and $x \notin G$ and so $x \notin \wedge_\kappa(A)$ which proves (a).

(b). The proof follows from the definition of $\wedge_\kappa$.

(c). By (b), $A \subset \wedge_\kappa(A)$ and so by (a), $\wedge_\kappa(A) \subset \wedge_\kappa(\wedge_\kappa(A))$. Let $x \notin \wedge_\kappa(A)$. then there exists $G \in \kappa$ such that $A \subset G$ and $x \notin G$ which implies that $\wedge_\kappa(A) \subset G$ and $x \notin G$. Therefore, $x \notin \wedge_\kappa(\wedge_\kappa(A))$ which implies that $\wedge_\kappa(\wedge_\kappa(A)) \subset \wedge_\kappa(A)$. This completes the proof.

(d) Clearly, by (a), $\cup\{\wedge_\kappa(C_\iota) \mid \iota \in \Delta\} \subset \wedge_\kappa(\cup\{C_\iota \mid \iota \in \Delta\})$. Conversely, suppose $x \notin \cup\{\wedge_\kappa(C_\iota) \mid \iota \in \Delta\}$. Then $x \notin \wedge_\kappa(C_\iota)$ for every $\iota \in \Delta$. Therefore, for every $\iota \in \Delta$, there exists $G_\iota \in \kappa$ such that $C_\iota \subset G_\iota$ and $x \notin G_\iota$. Let $G = \cup\{G_\iota \mid \iota \in \Delta\}$. Then $x \notin G$ and $\cup\{C_\iota \mid \iota \in \Delta\} \subset G$ which implies that $x \notin \wedge_\kappa(\cup\{C_\iota \mid \iota \in \Delta\})$. This completes the proof.

(e) The proof follows from (a).

(f) The proof follows from the definition of $\wedge_\kappa$.

(g) Let $x \in \wedge_\kappa(A)$. If $c_\kappa(\{x\}) \cap A = \emptyset$, then $X - c_\kappa(\{x\})$ is a $\kappa-$open set such that $A \subset X - c_\kappa(\{x\})$ and $x \notin X - c_\kappa(\{x\})$. Therefore, $x \notin \wedge_\kappa(A)$, a contradiction to the assumption and so $c_\kappa(\{x\}) \cap A \neq \emptyset$. Hence $\wedge_\kappa(A) \subset \{x \mid c_\kappa(\{x\}) \cap A \neq \emptyset\}$. Conversely, suppose for $x \in X,\ c_\kappa(\{x\}) \cap A \neq \emptyset$. If $x \notin \wedge_\kappa(A)$, then there exists a $\kappa-$open set G such that $A \subset G$ and $x \notin G$. Therefore, $x \in X - G$ which implies that $c_\kappa(\{x\}) \subset c_\kappa(X - G) = X - G \subset X - A$ and so $c_\kappa(\{x\}) \cap A = \emptyset$, a contradiction. Therefore, $\{x \mid c_\kappa(\{x\}) \cap A \neq \emptyset\} \subset \wedge_\kappa(A)$. This completes the proof.

(h) Suppose $y \in \wedge_\kappa(\{x\})$. Then $y \in G$ whenever G is a $\kappa-$open set containing x. Suppose $x \notin c_\kappa(\{y\})$, then there is a $\kappa-$closed set F such that $\{y\} \subset F$ and $x \notin F$. Since $X - F$ is a $\kappa-$open set containing x, $y \in F$ and so $c_\kappa(\{y\}) \subset c_\kappa(F) = F$ which implies that $c_\kappa(\{y\}) \cap \{x\} = \emptyset$. By (g), $y \notin \wedge_\kappa(\{x\})$, a contradiction. Hence $x \in c_\kappa(\{y\})$.

Conversely, suppose $x \in c_\kappa(\{y\})$. If $y \notin \wedge_\kappa(\{x\})$, there exists a $\kappa-$open set G containing x such that $y \notin G$. Now $y \in X - G$ implies that $c_\kappa(\{y\}) \subset c_\kappa(X - G) = X - G \subset X - \{x\}$ and so $c_\kappa(\{y\}) \cap \{x\} = \emptyset$ which implies that $x \notin c_\kappa(\{y\})$, a contradiction to the hypothesis. Therefore, $y \in \wedge_\kappa(\{x\})$. This completes the proof.

(i) Suppose $\wedge_\kappa(\{x\}) \neq \wedge_\kappa(\{y\})$. Assume that $z \in \wedge_\kappa(\{x\})$ and $z \notin \wedge_\kappa(\{y\})$. Then by (h) and (g), $x \in c_\kappa(\{z\})$ and $\{y\} \cap c_\kappa(\{z\}) = \emptyset$ and so $c_\kappa(\{x\}) \subset c_\kappa(\{z\})$ and $\{y\} \cap c_\kappa(\{z\}) = \emptyset$. Therefore, $c_\kappa(\{x\}) \cap \{y\} = \emptyset$ which implies that $c_\kappa(\{x\}) \neq c_\kappa(\{y\})$.

Conversely, suppose $c_\kappa(\{x\}) \neq c_\kappa(\{y\})$. Assume that $z \in c_\kappa(\{x\})$ and $z \notin c_\kappa(\{y\})$. By (h), $x \in \wedge_\kappa(\{z\})$ and $y \notin \wedge_\kappa(\{z\})$ and so $\wedge_\kappa(\{x\}) \subset \wedge_\kappa(\{z\})$ and $\{y\} \cap \wedge_\kappa(\{z\}) = \emptyset$ which implies that $\{y\} \cap \wedge_\kappa(\{x\}) = \emptyset$. Therefore, $\wedge_\kappa(\{y\}) \neq \wedge_\kappa(\{x\})$ which completes the proof.

Example 3.2. Let $X = \{a, b\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \emptyset,\ \gamma(\{a\}) = \{b\},\ \gamma(\{b\}) = \{b\},\ \gamma(X) = X$. Then $\mu = \{\emptyset, \{b\}, X\}$. If $A = \{a\},\ B = \{b\}$, then $\wedge_\mu(A) = X,\ \wedge_\mu(B) = B$ and $\wedge_\mu(A \cap B) = \wedge_\mu(\emptyset) = \emptyset$. Since $\wedge_\mu(A) \cap \wedge_\mu(B) = B,\ \wedge_\mu(A) \cap \wedge_\mu(B) \neq \wedge_\mu(A \cap B)$.

The proof of the following Theorem 3.3 is similar to that of Theorem 3.1 and hence the proof is omitted. Example 3.4 shows that the two sets in 3.3(d) are not equal. Theorem 3.5

below gives the relation between the operators $\wedge_\kappa$ and $\vee_\kappa$.

Theorem 3.3. Let A, B and $\{C_\iota \mid \iota \in \Delta\}$ be subsets of X, $\gamma \in \Gamma$ and $\kappa \in \Omega$. Then the following hold.

(a) If $A \subset B$, then $\vee_\kappa(A) \subset \vee_\kappa(B)$.

(b)$\vee_\kappa(A) \subset A$.

(c)$\vee_\kappa(\vee_\kappa(A)) = \vee_\kappa(A)$.

(d)$\vee_\kappa(\cup\{C_\iota \mid \iota \in \Delta\}) \supset \cup\{\vee_\kappa(C_\iota) \mid \iota \in \Delta\}$.

(e)$\vee_\kappa(\cap\{C_\iota \mid \iota \in \Delta\}) = \cap\{\vee_\kappa(C_\iota) \mid \iota \in \Delta\}$.

(f) If A is $\kappa-$closed, then $\vee_\kappa(A) = A$.

Example 3.4. Let $X = \{a, b, c, d\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \emptyset$, $\gamma(\{a\}) = \{b\}$, $\gamma(\{b\}) = \{c\}$, $\gamma(\{c\}) = \{b\}, \gamma(\{d\}) = \{d\}$, $\gamma(\{a, b\}) = \{b, c\}$, $\gamma(\{a, c\}) = \{b\}, \gamma(\{a, d\}) = \{b, d\}, \gamma(\{b, c\}) = \{b, c\}$, $\gamma(\{b, d\}) = \{c, d\}$, $\gamma(\{c, d\}) = \{b, d\}$, $\gamma(\{a, b, c\}) = \{b, c\}$, $\gamma(\{b, c, d\}) = \{b, c, d\}, \gamma(\{a, c, d\}) = X$, $\gamma(\{a, b, d\}) = \{b, c, d\}$ and $\gamma(X) = X$. Then $\mu = \{\emptyset, \{d\}, \{b, c\}, \{a, c, d\}, \{b, c, d\}, X\}$. If $A = \{a\}$, $B = \{d\}$, then $\vee_\mu(A) = \{a\}$, $\vee_\mu(B) = \emptyset$ and $\vee_\mu(A \cup B) = \vee_\mu(\{a, d\}) = \{a, d\}$. Since $\vee_\mu(A) \cup \vee_\mu(B) = A$, $\vee_\mu(A) \cup \vee_\mu(B) \neq \vee_\mu(A \cup B)$. If $A = \{a, c, d\}$, then $\vee_\mu(A) = A$ but A is not $\mu-$closed. This shows that the reverse direction of Theorem 3.3 (f) is not true.

If $A = \{c, d\}$, then $\wedge_\mu(A) = A$ but A is not $\mu-$open. This shows that the reverse direction of Theorem 3.1(f) is not true.

Theorem 3.5. Let A be a subset of X, $\gamma \in \Gamma$ and $\kappa \in \Omega$. Then the following hold.

(a)$\wedge_\kappa(X - A) = X - \vee_\kappa(A)$.

(b)$\vee_\kappa(X - A) = X - \wedge_\kappa(A)$.

(c) $(\wedge_\kappa)^\star = \vee_\kappa$.

(d)$(\vee_\kappa)^\star = \wedge_\kappa$.

Proof. (a) and (b) follow from the definitions of $\wedge_\kappa$ and $\vee_\kappa$.
(c) If $A \subset X$, then $(\wedge_\kappa)^\star(A) = X - \wedge_\kappa(X - A) = X - (X - \vee_\kappa(A)) = \vee_\kappa(A)$ and so $(\wedge_\kappa)^\star = \vee_\kappa$.
(d) The proof is similar to the proof of (c).

If X is a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$, a subset A of X is said to be a $\vee_\kappa-$*set* if $A = \vee_\kappa(A)$ and A is said to be a $\wedge_\kappa-$*set* if $A = \wedge_\kappa(A)$. In any space (X, κ), the following Theorem 3.6 lists out the $\vee_\kappa-$sets and the $\wedge_\kappa-$sets.

Theorem 3.6. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then the following hold.

(a) $\emptyset$ is a $\wedge_\kappa-$set.

(b) A is a $\wedge_\kappa-$set if and only if $X - A$ is a $\vee_\kappa-$set.

(c) X is a $\vee_\kappa-$set.

(d) The union of $\wedge_\kappa-$sets is again a $\wedge_\kappa-$set.

(e) The union of $\vee_\kappa-$sets is again a $\vee_\kappa-$set.

(f) The intersection of $\wedge_\kappa-$sets is again a $\wedge_\kappa-$set.

(g) The intersection of $\vee_\kappa-$sets is again a $\vee_\kappa-$set.

(h) If $\kappa \in \Omega_1 = \Omega - \{\mu,\ \alpha,\ \pi\}$, then X is a $\wedge_\kappa-$set and so $\emptyset$ is a $\vee_\kappa-$set.

Proof.

(a) follows from Theorem 3.1(f) since $\emptyset \in \kappa$ for every $\kappa \in \Omega$.

(b) Suppose A is a $\wedge_\kappa-$set. Then $A = \wedge_\kappa(A)$. Now $X - A = X - \wedge_\kappa(A) = \vee_\kappa(X - A)$, by Theorem 3.5(b). Therefore, $X - A$ is a $\vee_\kappa-$set. The proof of the converse is similar with follows from Theorem 3.5(a).

(c) follows from (a) and (b).

(d) Let $\{A_\iota \mid \iota \in \Delta\}$ be a family of $\wedge_\kappa-$sets. Therefore, $A_\iota = \wedge_\kappa(A_\iota)$ for every $\iota \in \Delta$. Now $\wedge_\kappa(\cup\{A_\iota \mid \iota \in \Delta\}) = \cup\{\wedge_\kappa(A_\iota) \mid \iota \in \Delta\}$, by Theorem 3.1(d) and so $\wedge_\kappa(\cup\{A_\iota \mid \iota \in \Delta\}) = \cup\{A_\iota \mid \iota \in \Delta\}$.

(e) Let $\{A_\iota \mid \iota \in \Delta\}$ be a family of $\vee_\kappa-$sets. Therefore, $A_\iota = \vee_\kappa(A_\iota)$ for every $\iota \in \Delta$. Now $\cup\{A_\iota \mid \iota \in \Delta\} = \cup\{\vee_\kappa(A_\iota) \mid \iota \in \Delta\} \subset \vee_\kappa(\cup\{A_\iota \mid \iota \in \Delta\})$ by Theorem 3.3(d) and so $\vee_\kappa(\cup\{A_\iota \mid \iota \in \Delta\}) = \cup\{A_\iota \mid \iota \in \Delta\}$ by Theorem 3.3(b).

(f) Let $\{A_\iota \mid \iota \in \Delta\}$ be a family of $\wedge_\kappa-$sets. Therefore, $A_\iota = \wedge_\kappa(A_\iota)$ for every $\iota \in \Delta$. Now $\cap\{A_\iota \mid \iota \in \Delta\} = \cap\{\wedge_\kappa(A_\iota) \mid \iota \in \Delta\} \supset \wedge_\kappa(\cap\{A_\iota \mid \iota \in \Delta\})$ by Theorem 3.1(e) and so $\wedge_\kappa(\cap\{A_\iota \mid \iota \in \Delta\}) = \cap\{A_\iota \mid \iota \in \Delta\}$ by Theorem 3.1(b).

(g) Let $\{A_\iota \mid \iota \in \Delta\}$ be a family of $\vee_\kappa-$sets. Therefore, $A_\iota = \vee_\kappa(A_\iota)$ for every $\iota \in \Delta$. Now $\vee_\kappa(\cap\{A_\iota \mid \iota \in \Delta\}) = \cap\{\vee_\kappa(A_\iota) \mid \iota \in \Delta\}$, by Theorem 3.3(e) and so $\vee_\kappa(\cap\{A_\iota \mid \iota \in \Delta\}) = \cap\{A_\iota \mid \iota \in \Delta\}$.

(h) Since $X \in \sigma$ by Lemma 2.2, $X \in \kappa$ for every $\kappa \in \Omega_1$ and so the proof follows from (a) and (b).

Remark 3.7. Let $\tau_\kappa = \{A \subset X \mid A = \wedge_\kappa(A)\}$ and $\tau^\kappa = \{A \subset X \mid A = \vee_\kappa(A)\}$. Then τ_κ and τ^κ are topologies by Theorem 3.6, such that arbitrary intersection of $\tau_\kappa-$open sets is a $\tau_\kappa-$open set and an arbitrary intersection of $\tau^\kappa-$open sets is a $\tau^\kappa-$open set.

Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. A subset A of X is called a generalized $\wedge_\kappa-$set (in short, $g.\wedge_\kappa-$set) if $\wedge_\kappa(A) \subset F$ whenever $A \subset F$ and F is $\kappa-$closed. B is called a generalized $\vee_\kappa-$set (in short, $g.\vee_\kappa-$set) if $X - B$ is a $g.\wedge_\kappa-$set. We will denote the family of all $g.\wedge_\kappa-$sets by $\mathbf{D}^{\wedge_\kappa}$ and the family of all $g.\vee_\kappa-$sets by $\mathbf{D}^{\vee_\kappa}$. The following Theorem 3.8 shows that $\mathbf{D}^{\wedge_\kappa}$ is closed under arbitrary union and $\mathbf{D}^{\vee_\kappa}$ is closed under arbitrary intersection. Theorem 3.9 below gives a characterization of $g.\vee_\kappa-$sets.

Theorem 3.8. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then the following hold.

(a) If $B_\iota \in \mathbf{D}^{\wedge_\kappa}$ for every $\iota \in \Delta$, then $\cup\{B_\iota \mid \iota \in \Delta\} \in \mathbf{D}^{\wedge_\kappa}$.

(b) If $B_\iota \in \mathbf{D}^{\vee_\kappa}$ for every $\iota \in \Delta$, then $\cap\{B_\iota \mid \iota \in \Delta\} \in \mathbf{D}^{\vee_\kappa}$.

Proof. (a) Let $B_\iota \in \mathbf{D}^{\wedge_\kappa}$ for every $\iota \in \Delta$. Then each B_ι is a $g.\wedge_\kappa-$set. Suppose F is $\kappa-$closed and $\cup\{B_\iota \mid \iota \in \Delta\} \subset F$. Then for every $\iota \in \Delta$, $B_\iota \subset F$ and F is $\kappa-$closed. By hypothesis, for every $\iota \in \Delta$, $\wedge_\kappa(B_\iota) \subset F$ and so $\cup\{\wedge_\kappa(B_\iota) \mid \iota \in \Delta\} \subset F$. By Theorem 3.1(d), $\wedge_\kappa(\cup\{B_\iota \mid \iota \in \Delta\}) \subset F$ and so $\cup\{B_\iota \mid \iota \in \Delta\} \in \mathbf{D}^{\wedge_\kappa}$.

(b) Let $B_\iota \in \mathbf{D}^{\vee_\kappa}$ for every $\iota \in \Delta$. Then each B_ι is a $g.\vee_\kappa-$set and so $X - B_\iota \in \mathbf{D}^{\wedge_\kappa}$ for every $\iota \in \Delta$. Now $X - (\cap\{B_\iota \mid \iota \in \Delta\}) = \cup(\{X - B_\iota \mid \iota \in \Delta\}) \in \mathbf{D}^{\wedge_\kappa}$, by (a). Therefore, $\cap\{B_\iota \mid \iota \in \Delta\} \in \mathbf{D}^{\vee_\kappa}$.

Theorem 3.9. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then a subset A of X is a $g.\vee_\kappa-$set if and only if $U \subset \vee_\kappa(A)$ whenever $U \subset A$ and U is $\kappa-$open.

Proof. Suppose A is a $g.\vee_\kappa-$set. Let U be a $\kappa-$open set such that $U \subset A$. Then $X - U$ is a $\kappa-$closed set such that $X - U \supset X - A$ and so $\wedge_\kappa(X - U) \supset \wedge_\kappa(X - A)$. Therefore, $X - U \supset \wedge_\kappa(X - A) = X - \vee_\kappa(A)$ and so $U \subset \vee_\kappa(A)$. Conversely, suppose the condition holds.

Let A be a subset of X. Let F be a $\kappa-$closed subset of X such that $X-A \subset F$. Then $X-F \subset A$ and so by hypothesis, $X - F \subset \vee_\kappa(A)$. Then $X - \vee_\kappa(A) \subset F$ and so $\wedge_\kappa(X - A) \subset F$ which implies that $X - A$ is a $g.\wedge_\kappa-$set. Therefore, A is a $g.\vee_\kappa-$set.

The remaining theorems in this section give some properties of $g.\vee_\kappa-$sets and $g.\wedge_\kappa-$sets.

Theorem 3.10. Let $x \in X$, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then the following hold.

(a) $\{x\}$ is either a $\kappa-$open set or $X - \{x\}$ is a $g.\wedge_\kappa-$set.

(b) $\{x\}$ is either a $\kappa-$open set or a $g.\vee_\kappa-$set.

Proof. (a) Suppose $\{x\}$ is not a $\kappa-$open set. Then X is the only $\kappa-$closed set containing $X - \{x\}$ and so $\wedge_\kappa(X - \{x\}) \subset X$. Therefore, $X - \{x\}$ is a $g.\wedge_\kappa-$set.

(b) Suppose $\{x\}$ is not a $\kappa-$open set. By (a), $X - \{x\}$ is a $g.\wedge_\kappa-$set and so $\{x\}$ is a $g.\vee_\kappa-$set.

Theorem 3.11. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. If B is a $g.\vee_\kappa-$set and F is a $\kappa-$closed set such that $\vee_\kappa(B) \cup (X - B) \subset F$, then $F = X$.

Proof. Since B is a $g.\vee_\kappa-$set, $X - B$ is a $g.\wedge_\kappa-$set such that $X - B \subset F$. Therefore, $\wedge_\kappa(X-B) \subset F$ which implies that $X-F \subset \vee_\kappa(B)$. Also, $\vee_\kappa(B) \subset F$ and so $X-F \subset X-\vee_\kappa(B)$. Hence $X - F \subset \vee_\kappa(B) \cap (X - \vee_\kappa(B)) = \emptyset$. Therefore, $F = X$.

Corollary 3.12. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. If B is a $g.\vee_\kappa-$set such that $\vee_\kappa(B) \cup (X - B)$ is $\kappa-$closed, then B is a $\vee_\kappa-$set.

Proof. By Theorem 3.11, $\vee_\kappa(B) \cup (X - B) = X$ and so $X - (\vee_\kappa(B) \cup (X - B)) = \emptyset$ which implies that $(X - \vee_\kappa(B)) \cap B = \emptyset$. Therefore, $B \subset \vee_\kappa(B)$ and so by Theorem 3.3(b), $B = \vee_\kappa(B)$.

Theorem 3.13. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. If A and B are subsets of X such that $A \subset B \subset \wedge_\kappa(A)$ and A is a $g.\wedge_\kappa-$set, then B is a $g.\wedge_\kappa-$set. In particular, if A is a $g.\wedge_\kappa-$set, then $\wedge_\kappa(A)$ is a $g.\wedge_\kappa-$set.

Proof. Since $A \subset B \subset \wedge_\kappa(A)$, $\wedge_\kappa(A) \subset \wedge_\kappa(B) \subset \wedge_\kappa(\wedge_\kappa(A)) = \wedge_\kappa(A)$ and so $\wedge_\kappa(A) = \wedge_\kappa(B)$. If F is any $\kappa-$closed set such that $B \subset F$, then $A \subset F$ and so $\wedge_\kappa(B) = \wedge_\kappa(A) \subset F$. Therefore, B is a $g.\wedge_\kappa-$set.

Corollary 3.14. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. If A and B are subsets of X such that $\vee_\kappa(A) \subset B \subset A$ and A is a $g.\vee_\kappa-$set, then B is a $g.\vee_\kappa-$set. In particular, if A is a $g.\vee_\kappa-$set, then $\vee_\kappa(A)$ is a $g.\vee_\kappa-$set.

§4. Some separation axioms in generalized topological spaces

Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then (X, κ) is called a $\kappa - T_1$ space if for all $x,\ y \in X,\ x \neq y$, there exists a $\kappa-$open set G such that $x \in G$ and $y \notin G$ and there exists a $\kappa-$open set H such that $x \notin H$ and $y \in H$. $\sigma - T_1$ space is defined in [5]. The following Theorem 4.1 gives a characterization of $\kappa - T_1$ spaces in terms of $\kappa-$closed sets and Theorem 4.2 gives a characterization of $\kappa - T_1$ spaces in terms of $\wedge_\kappa-$sets.

Theorem 4.1. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then (X, κ) is a $\kappa - T_1$ space if and only if every singleton set is a $\kappa-$closed set.

Proof. Suppose (X, κ) is a $\kappa - T_1$ space. Let $x \in X$. If $y \in X - \{x\}$, then $x \neq y$. By hypothesis, there exists a $\kappa-$open set H_y such that $y \in H_y$ and $x \notin H_y$ and so $y \in H_y \subset X-\{x\}$.

Hence $X-\{x\}=\cup\{H_y \mid y\in X-\{x\}\ \}$ is $\kappa-$open and so $\{x\}$ is $\kappa-$closed. Conversely, suppose each singleton set is a $\kappa-$closed set. Let $x,y\in X$ such that $x\neq y$. Then $X-\{x\}$ and $X-\{y\}$ are $\kappa-$open sets such that $y\in X-\{x\}$ and $x\in X-\{y\}$. Therefore, (X,κ) is a $\kappa-T_1$ space.

Theorem 4.2. Let X be a nonempty set, $\gamma\in\Gamma(X)$ and $\kappa\in\Omega$. Then (X,κ) is a $\kappa-T_1$ space if and only if every subset of X is a $\wedge_\kappa-$set.

Proof. Suppose (X,κ) is a $\kappa-T_1$ space. Let A be subset of X. By Theorem 3.1(b), $A\subset\wedge_\kappa(A)$. Suppose $x\notin A$. Then $X-\{x\}$ is a $\kappa-$open set such that $A\subset X-\{x\}$ and so $\wedge_\kappa(A)\subset X-\{x\}$. Hence every subset of X is a $\wedge_\kappa-$set. Conversely, suppose every subset of X is a $\wedge_\kappa-$set and so $\wedge_\kappa(\{x\})=\{x\}$ for every $x\in X$. Let $x,\ y\in X$ such that $x\neq y$. Then $y\notin\wedge_\kappa(\{x\})$ and $x\notin\wedge_\kappa(\{y\})$. Since $y\notin\wedge_\kappa(\{x\})$, there is a $\kappa-$open set U such that $x\in U$ and $y\notin U$. Similarly, since $x\notin\wedge_\kappa(\{y\})$, there is a $\kappa-$open set V such that $y\in V$ and $x\notin V$. Therefore, (X,κ) is a $\kappa-T_1$ space.

Corollary 4.3. Let X be a nonempty set, $\gamma\in\Gamma(X)$ and $\kappa\in\Omega$. Then the following are equivalent.

(a) (X,κ) is a $\kappa-T_1$ space.

(b) Every subset of X is a $\wedge_\kappa-$set.

(c) Every subset of X is a $\vee_\kappa-$set.

Proof. (a) and (b) are equivalent by Theorem 4.2.
(b) and (c) are equivalent by Theorem 3.5(b).

Theorem 4.4. Let X be a nonempty set, $\gamma\in\Gamma(X),\ \kappa\in\Omega$ and $A\subset X$. Then the following hold.

(a) If A is a $\wedge_\kappa-$set, then A is a $g.\wedge_\kappa-$set.

(b) If A is a $\vee_\kappa-$set, then A is a $g.\vee_\kappa-$set. The reverse directions are true if (X,κ) is a $\kappa-T_1$ space.

Proof.

(a) Suppose A is a $\wedge_\kappa-$set. Then $A=\wedge_\kappa(A)$. If $A\subset F$ where F is $\kappa-$closed, then $A=\wedge_\kappa(A)\subset F$. Therefore, A is a $g.\wedge_\kappa-$set.

(b) The proof is similar to the proof of (a).

The reverse directions follow from Corollary 4.3.

Let X be a nonempty set, $\gamma\in\Gamma(X)$ and $\kappa\in\Omega$. (X,κ) is said to be a $\kappa-T_0$ space if for distinct points x and y in X, there exists a $\kappa-$open set G containing one but not the other. Clearly, every $\kappa-T_1$ space is a $\kappa-T_0$ space. Since every topology is a generalized topology and in a topological space, T_0 spaces need not be T_1 spaces, $\kappa-T_0$ spaces need not be $\kappa-T_0$ spaces. Theorem 4.5 gives a characterization of $\kappa-T_0$ spaces. The easy proof of Corollary 4.6 is omitted.

Theorem 4.5. Let (X,κ) be a $\kappa-$space where $\gamma\in\Gamma(X)$ and $\kappa\in\Omega$. Then (X,κ) is a $\kappa-T_0$ space if and only if distinct points of X have distinct $\kappa-$closures.

Proof. Suppose (X,κ) is a $\kappa-T_0$ space. Let x and y be points of X such that $x\neq y$. Then there exists a $\kappa-$open set G containing one but not the other, say $x\in G$ and $y\notin G$. Then $y\notin c_\kappa(\{x\})$ and so $c_\kappa(\{x\})\neq c_\kappa(\{y\})$. Conversely, suppose distinct points of X have distinct $\kappa-$closures. Let x and y be points of X such that $x\neq y$. Then $c_\kappa(\{x\})\neq c_\kappa(\{y\})$. Suppose $z\in c_\kappa(\{x\})$ and $z\notin c_\kappa(\{y\})$. If $x\in c_\kappa(\{y\})$, then $c_\kappa(\{x\})\subset c_\kappa(\{y\})$ and so $z\in c_\kappa(\{y\})$, a

contradiction. Therefore, $x \notin c_\kappa(\{y\})$ which implies that $x \in X - c_\kappa(\{y\})$ and $X - c_\kappa(\{y\})$ is $\kappa-$open. Hence (X, κ) is a $\kappa - T_0$ space.

Corollary 4.6. Let (X, κ) be a $\kappa-$space where $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then (X, κ) is a $\kappa - T_0$ space if and only if for distinct points x and y of X, either $x \notin c_\kappa(\{y\})$ or $y \notin c_\kappa(\{x\})$.

Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. (X, κ) is said to be a $\kappa - T_2$ space if for distinct points x and y in X, there exist disjoint $\kappa-$open sets G and H such that $x \in G$ and $y \in H$. Clearly, every $\kappa - T_2$ space is a $\kappa - T_1$ space and the converse is not true. Theorem 4.7 below gives characterizations of $\kappa - T_2$ spaces.

Theorem 4.7. In a $\kappa-$space (X, κ), where $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$, the following statements are equivalent.

(a) (X, κ) is a $\kappa - T_2$ space.

(b) For each $x \in X$ and $y \neq x$, there exists a $\kappa-$open set U such that $x \in U$ and $y \notin c_\kappa(U)$.

(c) For every $x \in X$, $\{x\} = \cap\{c_\kappa(U) \mid x \in U \text{ and } U \text{ is } \kappa - open\}$.

Proof. (a)$\Rightarrow$(b). Let $x,\ y \in X$ such that $y \neq x$. Then, there exists disjoint $\kappa-$open sets U and H such that $x \in U$ and $y \in H$. Then $X - H$ is a $\kappa-$closed set such that $U \subset X - H$ and so $c_\kappa(U) \subset X - H$. U is the required $\kappa-$open set such that $x \in U$ and $y \notin c_\kappa(U)$.
(b)$\Rightarrow$(c). Let $x \in X$. If $y \in X$ such that $x \neq y$, by (b), there exists a $\kappa-$open set U such that $x \in U$ and $y \notin c_\kappa(U)$. Clearly, $\{x\} = \cap\{c_\kappa(U) \mid x \in U \text{ and } U \text{ is } \kappa - open\}$.
(c)$\Rightarrow$(a). Let $x,\ y \in X$ such that $y \neq x$. Then $y \notin \{x\} = \cap\{c_\kappa(U) \mid x \in U \text{ and } U \text{ is } \kappa\text{−open}\}$, by (c). Therefore, $y \notin c_\kappa(U)$ for some $\kappa-$open set containing x. U and $X - c_\kappa(U)$ are the required disjoint $\kappa-$open sets containing x and y respectively.

Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then (X, κ) is said to be a $\kappa - R_0$ space if every $\kappa-$open subset of X contains the $\kappa-$closure of its singletons. (X, κ) is said to be a $\kappa - R_1$ space if for $x, y \in X$ with $c_\kappa(\{x\}) \neq c_\kappa(\{y\})$, there exist disjoint $\kappa-$open sets G and H such that $c_\kappa(\{x\}) \subset G$ and $c_\kappa(\{y\}) \subset H$. Clearly, every $\kappa - R_1$ space is a $\kappa - R_0$ space but the converse is not true. The following Theorem 4.8 follows from Theorem 4.1. Theorem 4.9 below gives a characterization of $\kappa - R_0$ spaces.

Theorem 4.8. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then every $\kappa - T_1$ space is a $\kappa - R_0$ space.

Theorem 4.9. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then (X, κ) is a $\kappa - R_0$ space if and only if every $\kappa-$open subset of X is the union of $\kappa-$closed sets.

Proof. Suppose (X, κ) is a $\kappa - R_0$ space. If A is $\kappa-$open, then for each $x \in A, c_\kappa(\{x\}) \subset A$ and so $\cup\{cl_\kappa\{x\} \mid x \in A\} \subset A$. It follows that $A = \cup\{cl_\kappa\{x\} \mid x \in A\}$. Conversely, suppose A is $\kappa-$open and $x \in A$. Then by hypothesis, $A = \cup\{B_\iota \mid \iota \in \Delta\}$ where each B_ι is $\kappa-$closed. Now $x \in A$, implies that $x \in B_\iota$ for some $\iota \in \Delta$. Therefore, $c_\kappa(\{x\}) \subset B_\iota \subset A$ and so (X, κ) is a $\kappa - R_0$ space.

Theorem 4.10. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then (X, κ) is a $\kappa - R_0$ space if and only if for $x,\ y \in X$, $c_\kappa(\{x\}) \neq c_\kappa(\{y\})$ implies that $c_\kappa(\{x\}) \cap c_\kappa(\{y\}) = \emptyset$.

Proof. Suppose (X, κ) is a $\kappa - R_0$ space. Let $x, y \in X$, such that $c_\kappa(\{x\}) \neq c_\kappa(\{y\})$. Suppose $z \in c_\kappa(\{x\})$ and $z \notin c_\kappa(\{y\})$. Since $z \notin c_\kappa(\{y\})$, there exists a $\kappa-$open set G containing z such that $y \notin G$. Since $z \in c_\kappa(\{x\})$, $x \in G$. Since $y \notin G$, it follows that $x \notin c_\kappa(\{y\})$ and so $x \in X - c_\kappa(\{y\})$. By hypothesis, $c_\kappa(\{x\}) \subset X - c_\kappa(\{y\})$ and so $c_\kappa(\{x\}) \cap c_\kappa(\{y\}) = \emptyset$. Conversely,

suppose the condition holds. Let G be a $\kappa-$open set such that $x \in G$. If $y \notin G$, then $x \neq y$ and so $x \notin c_\kappa(\{y\})$ which implies that $c_\kappa(\{x\}) \neq c_\kappa(\{y\})$. By hypothesis, $c_\kappa(\{x\}) \cap c_\kappa(\{y\}) = \emptyset$ and so $y \notin c_\kappa(\{x\})$. Hence $c_\kappa(\{x\}) \subset G$ which implies that (X, κ) is a $\kappa - R_0$ space.

Theorem 4.11. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then (X, κ) is a $\kappa - R_0$ space if and only if for $x,\ y \in X,\ \wedge_\kappa(\{x\}) \neq \wedge_\kappa(\{y\})$ implies that $\wedge_\kappa(\{x\}) \cap \wedge_\kappa(\{y\}) = \emptyset$.

Proof. Suppose (X, κ) is a $\kappa - R_0$ space. Let $x,\ y \in X$ such that $\wedge_\kappa(\{x\}) \neq \wedge_\kappa(\{y\})$. Suppose that $z \in \wedge_\kappa(\{x\}) \cap \wedge_\kappa(\{y\})$. Then $z \in \wedge_\kappa(\{x\})$ and $z \in \wedge_\kappa(\{y\})$. By Theorem 3.1(h), $x \in c_\kappa(\{z\})$ and $y \in c_\kappa(\{z\})$ and so $c_\kappa(\{x\}) \cap c_\kappa(\{z\}) \neq \emptyset$ and $c_\kappa(\{y\}) \cap c_\kappa(\{z\}) \neq \emptyset$. By Theorem 4.10, $c_\kappa(\{x\}) = c_\kappa(\{z\})$ and $c_\kappa(\{y\}) = c_\kappa(\{z\})$ and so $c_\kappa(\{x\}) = c_\kappa(\{y\})$. By Theorem 3.1(i),$\wedge_\kappa(\{x\}) = \wedge_\kappa(\{y\})$, a contradiction. Therefore, $\wedge_\kappa(\{x\}) \cap \wedge_\kappa(\{y\}) = \emptyset$. Conversely, suppose the condition holds. Let $x, y \in X$ such that $c_\kappa(\{x\}) \neq c_\kappa(\{y\})$. Suppose that $z \in c_\kappa(\{x\}) \cap c_\kappa(\{y\})$. Then $z \in c_\kappa(\{x\})$ and $z \in c_\kappa(\{y\})$. By Theorem 3.1(h), $x \in \wedge_\kappa(\{z\})$ and $y \in \wedge_\kappa(\{z\})$ and so $\wedge_\kappa(\{x\}) \cap \wedge_\kappa(\{z\}) \neq \emptyset$ and $\wedge_\kappa(\{y\}) \cap \wedge_\kappa(\{z\}) \neq \emptyset$. By hypothesis, $\wedge_\kappa(\{x\}) = \wedge_\kappa(\{z\})$ and $\wedge_\kappa(\{y\}) = \wedge_\kappa(\{z\})$ and so $\wedge_\kappa(\{x\}) = \wedge_\kappa(\{y\})$. By Theorem 3.1(i), $c_\kappa(\{x\}) = c_\kappa(\{y\})$, a contradiction. Therefore, $c_\kappa(\{x\}) \cap c_\kappa(\{y\}) = \emptyset$. By Theorem 4.10, (X, κ) is a $\kappa - R_0$ space.

Theorem 4.12. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then the following are equal.

(a) (X, κ) is a $\kappa - R_0$ space.

(b) For any nonempty set A and a $\kappa-$open set G such that $A \cap G \neq \emptyset$, there exists a $\kappa-$closed set F such that $A \cap F \neq \emptyset$ and $F \subset G$.

(c) If G is $\kappa-$open, then $G = \cup\{F \mid F \subset G$ and F is $\kappa-$closed$\}$.

(d) If F is $\kappa-$closed, then $F = \cap\{G \mid F \subset G$ and G is $\kappa-$open$\}$.

(e) For every $x \in X,\ c_\kappa(\{x\}) \subset \wedge_\kappa(\{x\})$.

Proof. (a)$\Rightarrow$(b). Suppose (X, κ) is a $\kappa - R_0$ space. Let A be a nonempty set and G be a $\kappa-$open set such that $A \cap G \neq \emptyset$. If $x \in A \cap G$, then $x \in G$ and so by hypothesis, $c_\kappa(\{x\}) \subset G$. If $F = c_\kappa(\{x\})$, then F is the required $\kappa-$closed set such that $A \cap F \neq \emptyset$ and $F \subset G$.
(b)$\Rightarrow$(c). Let G be $\kappa-$open. Clearly, $G \supset \cup\{F \mid F \subset G$ and F is $\kappa-$closed$\}$. If $x \in G$, then $\{x\} \cap G \neq \emptyset$ and so by (b), there is a $\kappa-$closed set F such that $\{x\} \cap F \neq \emptyset$ and $F \subset G$ which implies that $x \in \{F \mid F \subset G$ and F is $\kappa-$closed$\}$. Therefore, $G \subset \{F \mid F \subset G$ and F is $\kappa-$closed$\}$. This completes the proof.
(c)$\Rightarrow$(d). Let F be $\kappa-$closed. By (c), $X - F = \cup\{K \mid K \subset X - F$ and K is $\kappa-$closed$\}$ and so $F = \cap\{X - K \mid F \subset X - K$ and $X - K$ is $\kappa-$open$\}$.
(d)$\Rightarrow$(e). Let $x \in X$. If $y \notin \wedge_\kappa(\{x\})$, then by Theorem 3.1(g), $\{x\} \cap c_\kappa(\{y\}) = \emptyset$. By (d), $c_\kappa(\{y\}) = \cap\{G \mid c_\kappa(\{y\}) \subset G$ and G is $\kappa-$open$\}$. Therefore, there is a $\kappa-$open G such that $c_\kappa(\{y\}) \subset G$ and $x \notin G$ which implies that $y \notin c_\kappa(\{x\})$. Therefore, $c_\kappa(\{x\}) \subset \wedge_\kappa(\{x\})$.
(e)$\Rightarrow$(a). Let G be a $\kappa-$open set such that $x \in G$. If $y \in c_\kappa(\{x\})$, then by (e), $y \in \wedge_\kappa(\{x\})$. Since $\wedge_\kappa(\{x\}) \subset \wedge_\kappa(G) = G,\ y \in G$. Hence (X, κ) is a $\kappa - R_0$ space.

Corollary 4.13. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then the following are equivalent.

(a) (X, κ) is a $\kappa - R_0$ space.

(b) For every $x \in X,\ c_\kappa(\{x\}) = \wedge_\kappa(\{x\})$.

Proof. (a)$\Rightarrow$(b). Let $x \in X$. By Theorem 4.12, $c_\kappa(\{x\}) \subset \wedge_\kappa(\{x\})$. To prove the direction, assume that $y \in \wedge_\kappa(\{x\})$. By Theorem 3.1(h), $x \in c_\kappa(\{y\})$ and so $c_\kappa(\{x\}) \subset c_\kappa(\{y\})$ which implies that $c_\kappa(\{x\}) \cap c_\kappa(\{y\}) \neq \emptyset$. By Theorem 4.10, $c_\kappa(\{x\}) = c_\kappa(\{y\})$ and so $y \in c_\kappa(\{x\})$. Hence $c_\kappa(\{x\}) = \wedge_\kappa(\{x\})$.
(b)$\Rightarrow$(a). The proof follows from Theorem 4.12.

Theorem 4.14. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then the following are equivalent.

(a) (X, κ) is a $\kappa - R_0$ space.

(b) For all $x,\ y \in X, x \in c_\kappa(\{y\})$ if and only if $y \in c_\kappa(\{x\})$.

Proof. (a)$\Rightarrow$(b). Let $x,\ y \in X$ such that $x \in c_\kappa(\{y\})$. By Corollary 4.13, $x \in \wedge_\kappa(\{y\})$ and so by Theorem 3.1(h), $y \in c_\kappa(\{x\})$. Thus $x \in c_\kappa(\{y\})$ implies that $y \in c_\kappa(\{x\})$. Similarly, we can prove that $y \in c_\kappa(\{x\})$ implies that $x \in c_\kappa(\{y\})$.
(b)$\Rightarrow$(a). Conversely, suppose the condition holds. Let $x \in X$. If $y \in c_\kappa(\{x\})$, then by hypothesis, $x \in c_\kappa(\{y\})$ and so by Theorem 3.1(h), $y \in \wedge_\kappa(\{x\})$ which implies that $c_\kappa(\{x\}) \subset \wedge_\kappa(\{x\})$. By Theorem 4.12(e), (X, κ) is a $\kappa - R_0$ space.

The following Theorem 4.15 gives a characterization of $\kappa - R_1$ space and Theorem 4.16 gives a characterization of $\kappa - T_2$ space.

Theorem 4.15. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then the following are equivalent.

(a) (X, κ) is a $\kappa - R_1$ space.

(b) For $x,\ y \in X$ such that $\wedge_\kappa(\{x\}) \neq \wedge_\kappa(\{y\})$, there exist disjoint $\kappa-$open sets G and H such that $c_\kappa(\{x\}) \subset G$ and $c_\kappa(\{y\}) \subset H$.

Proof. (a)$\Rightarrow$(b). Let $x,\ y \in X$ such that $\wedge_\kappa(\{x\}) \neq \wedge_\kappa(\{y\})$. Then by Corollary 4.13, since (X, κ) is a $\kappa - R_0$ space, $c_\kappa(\{x\}) \neq c_\kappa(\{y\})$ and so there exists disjoint $\kappa-$open sets G and H such that $c_\kappa(\{x\}) \subset G$ and $c_\kappa(\{y\}) \subset H$.
(b)$\Rightarrow$(a). Let $x,\ y \in X$ such that $c_\kappa(\{x\}) \neq c_\kappa(\{y\})$. By Theorem 3.1(i), $\wedge_\kappa(\{x\}) \neq \wedge_\kappa(\{y\})$. By hypothesis, there exist disjoint $\kappa-$open sets G and H such that $c_\kappa(\{x\}) \subset G$ and $c_\kappa(\{y\}) \subset H$ and so (X, κ) is a $\kappa - R_1$ space.

Theorem 4.16. Let X be a nonempty set, $\gamma \in \Gamma(X)$ and $\kappa \in \Omega$. Then the following are equivalent.

(a)(X, κ) is a $\kappa - T_2$ space.

(b)(X, κ) is both a $\kappa - R_1$ space and a $\kappa - T_1$ space.

(c)(X, κ) is both a $\kappa - R_1$ space and a $\kappa - T_0$ space.

Proof. (a)$\Rightarrow$(b). Suppose (X, κ) is a $\kappa - T_2$ space. Clearly, (X, κ) is a $\kappa - T_1$ space and so singletons are $\kappa-$closed sets, by Theorem 4.1. If $x,\ y \in X$ with $c_\kappa(\{x\}) \neq c_\kappa(\{y\})$, then $x \neq y$ and so there exist disjoint $\kappa-$open sets G and H such that $x \in G$ and $y \in H$. Therefore, $c_\kappa(\{x\}) \subset G$ and $c_\kappa(\{y\}) \subset H$ which implies that (X, κ) is a $\kappa - R_1$ space.
(b)$\Rightarrow$(c). The proof is clear.
(c)$\Rightarrow$(a). Let $x,\ y \in X$ such that $x \neq y$. Since (X, κ) is a $\kappa - T_0$ space, by Theorem 4.5, $c_\kappa(\{x\}) \neq c_\kappa(\{y\})$. Since (X, κ) is a $\kappa - R_1$ space, there exist disjoint $\kappa-$open sets G and H such that $c_\kappa(\{x\}) \subset G$ and $c_\kappa(\{y\}) \subset H$. Therefore, (X, κ) is a $\kappa - T_2$ space.

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Generalized galilean transformations and dual Quaternions

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Abstract Quaternion operators have an important role as a screw and a rotation operator in Euclidean motion. We introduce a new operator similar to quaternion operator in Galilean motion. This operator is defined as a dual quaternion operator by using some information in [1]. Then, we have geneneralized this operator for n-dimensional Galilean space.

Keywords Galilean transformation, dual complex numbers, Quaternion operators, lie group.

§1. Introduction

A unit real quaternion is a rotation operator on rigid body motion in Euclidean space. Unit dual quaternions are also used both as rotation and screw operators. Unit dual quaternions are seen as screw operator especially in Mechanics and Kinematics. Galilean geometry of motions was studied in [3]. n- dimensional dual complex numbers was given in [4]. These numbers are viewed as analysis. Galilean transformations are given as shear motion on plane [2]. Shear motion in Galiean space G_3 was given [2]. Moreover, union of shear motion and Euclidean motions was introduced. Galilean transformation (shear motion) was given by quaternions (in dual quaternion form) [1]. Here, we redefine dual quaternions in a new way for the first time. We work out Majernik's work in a new point of view by using structures of Lie groups and algebras. These are subgroups of Heisenberg Lie groups. We obtain elements of groups by the exponential expansion of quaternion forms of an element of Lie algebra. And we extended the work to the Galilean space G_n. Finally, we give Galilean transformation as dual quaternion operators.

§2. Galilean transformations in galilean space G_2

Galilean transformations were examined widely in [2]. Let $X \in R^n$ and G_n be Galilean space $(R^n, \|\|)$ with

$$\|X\| = \begin{cases} |x_1|, & x_1 \neq 0 \\ \sqrt{x_2^2 + x_3^2 + ... + x_n^2}, & x_1 = 0 \end{cases}$$

for $n = 2, 3, 4$. We redefine Galilean transformation by using quaternion operators. For any $X = (x, y) \in G_2$, Galilean transformation (shear motion) is defined as the following:

$$f : G_2 \to G_2\ , X \to f(X) = (x, \upsilon x + y),$$

f is a linear transformation, so f has the matrices form as the following

$$f(X) = \begin{bmatrix} 1 & 0 \\ \upsilon & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Lemma 2.1. Let f be a linear transformation. Then f is a Galilean transformation, where

$$f : G_2 \to G_2, X \to f(X) = (x, \upsilon x + y).$$

Proof. For $x \neq 0$ and $x = 0$, we have

$$\|X\| = |x| = \|f(X)\|$$

and

$$\|X\| = |y| = \|f(X)\|.$$

f is a Galilean transformation, because the linear function f is a isometry.

Theorem 2.1. Let $Gal(2)$ be a Lie group. Then $Gal(2)$ and $g(2)$ are Lie algebras of $Gal(2)$, where

$$Gal(2) = \left\{ \begin{bmatrix} 1 & 0 \\ \upsilon & 1 \end{bmatrix} : \upsilon \in R \right\}, g(2) = \left\{ \begin{bmatrix} 0 & 0 \\ \upsilon_1 & 0 \end{bmatrix} : \upsilon_1 \in R \right\}.$$

§3. Dual numbers and galilean transformation in G_2

Every vector $X = (x, y)$ in R^2 can be written as $X = x + \varepsilon y$ with $\varepsilon^2 = 0$. So $sp\{1, \varepsilon\} = R^2$. The form of $x + \varepsilon y$ is called dual quaternion form of X.

Lemma 3.1. Dual quaternion $Q = 1 + \varepsilon\upsilon$ is a Galilean transformation in G_2.

Proof. Since $Q = 1 + \varepsilon\upsilon$, we have

$$QX = (1 + \varepsilon\upsilon)(x + \varepsilon y) = x + \varepsilon(y + \upsilon x)$$

and

$$\|QX\| = \|X\|.$$

So, Q is a Galilean transformation in G_2. By using exponential map from Lie algebra to Lie group, on $g \in g(2)$ as in $g = \varepsilon\upsilon_1$ form :

$$e : g(2) \to Gal(2)$$
$$g \to e^g = e^{\varepsilon\upsilon_1} = 1 + \varepsilon\upsilon_1.$$

Corollary 3.1. $Q = e^g = 1 + \varepsilon\upsilon_1$ is a dual quaternion operator. Thus dual quaternion operator Q is a Galilean transformation.

§4. Galilean transformation(shear motion) in galilean space G_3

In this section shear motion on Galilean spaces G_3, Lie group structure of this motion and exponential form are given.

Theorem 4.1. f is a Galilean transformation(shear motion), where

$$\begin{aligned} f &: G_3 \to G_3 \\ X &\to f(X) = (x, ax+y, bx+z) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \end{aligned}$$

Proof. For $x \neq 0$ and $x = 0$, we have

$$\|f(X)\| = |x| = \|X\|$$

and

$$\|f(X)\| = \sqrt{y^2 + z^2} = \|X\|.$$

So, f is a Galilean transformation.

Theorem 4.2. $Gal(3)$ is a Lie group and the $g(3)$ is a Lie algebra of $Gal(3)$, where

$$Gal(3) = \left\{ A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} | a, b \in R \right\},$$

$$g(3) = \left\{ a = \begin{bmatrix} 0 & 0 & 0 \\ a_1 & 0 & 0 \\ b_1 & 0 & 0 \end{bmatrix} | a_1, b_1 \in R \right\}.$$

§5. Heisenberg lie group

The set of matrices

$$H = \left\{ \begin{bmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix} : x_i \in R, i = 1, 2, 3 \right\}$$

is a Lie group with repect to the matrix multiplication. This Heisenberg group has many important applications on Sub-Riemannian geometry and has very important role in physics.

Lemma 5.1. $Gal(3)$ is a subgroup of Heisenberg Lie group.

Proof. $Gal(3)$ is obtained from Heissenberg Lie group by taking $x_2 = 0$.

§6. Dual Quaternions and galilean transformation in G_3

Every vector X in R^3 can be written as $X = x + iy + jz$, where $i^2 = j^2 = ij = ji = 0$ and $sp\{1, i, j\} = R^3$. The form $X = x + iy + jz$ is called as dual quaternion form of $X \in R^3$.

Lemma 6.1. $Q = 1 + ai + bj$ is a Galilean transformation in G_3.

Proof. Since $Q = 1 + ai + bj$, we have

$$\begin{aligned} QX &= (1 + ai + bj)(x + iy + jz) \\ &= x + (ax + y)i + (bx + z)j \end{aligned}$$

and

$$\|QX\| = \|X\| .$$

Thus the Q is a Galilean transformation.Furthermore, we can write $g \in g(3)$ as $g = a_1 i + b_1 j$. So we can write an exponential map as follows:

$$e : g(3) \to Gal(3)$$

$$g \to e^g = e^{a_1 i + b_1 j} = 1 + a_1 i + b_1 j.$$

Corollary 6.1. $e^g = Q = 1 + a_1 i + b_1 j$ is a dual quaternion operator. Thus the dual quaternion operator Q is a Galilean transformation.

§7. Galilean transformations in galilean space G_4

In this part shear motion is defined in Galilean spaces G_4. Structure of Lie group of this motion and exponential form are given.

Theorem 7.1. f is a Galilean transformation(shear motion), where

$$f : G_4 \to G_4$$

$$\begin{aligned} X \to f(X) &= (x, ax + y, bx + z, cx + l) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ l \end{bmatrix} . \end{aligned}$$

Proof. For $x \neq 0$ and $x = 0$, we have

$$\|f(X)\| = |x| = \|X\|$$

and

$$\|f(X)\| = \sqrt{y^2 + z^2 + l^2} = \|X\| .$$

So, f is isometry and Galilean transformation.

Theorem 7.2. $Gal(4)$ is a Lie group and $g(4)$ is a Lie algebra of the $Gal(4)$, where

$$Gal(4)=\left\{A=\begin{bmatrix}1&0&0&0\\a&1&0&0\\b&0&1&0\\c&0&0&1\end{bmatrix}|a,b,c\in R\right\},$$

$$g(4)=\left\{a=\begin{bmatrix}0&0&0&0\\a_1&0&0&0\\b_1&0&0&0\\c_1&0&0&0\end{bmatrix}|a_1,b_1,c_1\in R\right\}.$$

§8. Dual Quaternions and galilean transformations in G_4

In this part, we reviewed study in [1]. Every vector X in R^3 can be written as the form $X=x+iy+jz+kl$, where

$$i^2=j^2=k^2=ij=ji=ik=ki=kj=jk=0$$

and

$$sp\{1,i,j,k\}=R^4.$$

The form $X=x+iy+jz+kl$ is called as dual quaternion form of $X\in R^4$.

Lemma 8.1. $Q=1+ai+bj+dk$ is a Galilean transformation in G_4.

Proof. Since $Q=1+ai+bj+dk$, we have

$$\begin{aligned}QX &= (1+ai+bj+dk)(x+iy+jz+kl)\\ &= x+(ax+y)i+(bx+z)j+(dx+l)k\end{aligned}$$

and

$$\|QX\|=\|X\|.$$

Thus, Q is a Galilean transformation. Furthermore, we can write $g\in g(3)$ as $g=a_1i+b_1j+d_1k$. So we can write an exponential map as follows:

$$e:g(4)\to Gal(4)$$

$$g\to e^g=e^{a_1i+b_1j+d_1k}=1+a_1i+b_1j+d_1k.$$

Corollary 8.1. $Q=e^g=1+a_1i+b_1j+d_1k$ is a dual quaternion operator. Thus dual quaternion operator Q is a Galilean transformation.

§9. Galilean transformations in galilean space G_n

In this part, Galilean transformation generalized to Galilean spaces G_n, by using Galilean transformation in spaces G_2, G_3 and G_4 .

Theorem 9.1. f is a Galilean transformation, where

$$f : G_n \to G_n$$

$$X \to f(X) = (x_1, \upsilon_1 x_1 + x_2, ..., \upsilon_{n-1} x_1 + x_n)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \upsilon_1 & 1 & 0 & 0 & 0 \\ \upsilon_2 & 0 & 1 & 0 & 0 \\ . & . & . & . & . \\ \upsilon_{n-1} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ x_n \end{bmatrix}.$$

Proof. For $x_1 \neq 0$ and $x_1 = 0$, we have

$$\|f(X)\| = |x_1| = \|X\|$$

and

$$\|f(X)\| = \sqrt{{x_2}^2 + {x_3}^2 + \cdots + {x_n}^2} = \|X\|.$$

Then, f is isometry, thus f is a Galilean transformation.

Theorem 9.2. $Gal(n)$ is a Lie group and $g(n)$ is a Lie algebra of $Gal(4)$, where

$$Gal(n) = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \upsilon_1 & 1 & 0 & 0 & 0 & 0 \\ \upsilon_2 & 0 & 1 & 0 & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \upsilon_{n-1} & 0 & 0 & 0 & 0 & 1 \end{bmatrix} | \upsilon_1, \upsilon_2, ..., \upsilon_{n-1} \in R \right\},$$

$$g(n) = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n-1} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} | a_1, a_2, \cdots, a_{n-1} \in R \right\}.$$

§10. Dual Quaternions and galilean transformation in G_n

In this part, every vector X in R^n is written in form $X = x_1 + x_2 i_1 + x_3 i_2 + \cdots + x_n i_{n-1}$, where $x_1, x_2, \cdots, x_n$ are real numbers. The components of X and $i_1, i_2, \cdots, i_{n-1}$ are units which satisfy the relations $i_1^2 = i_2^2 = \cdots = i_{n-1}^2 = 0$ and $i_j i_k = i_k i_j = 0$, $1 \leq k, j \leq n-1$. Also, $sp\{1,\ i_1, i_2, \cdots, i_{n-1}\} = R^n$, $X = x_1 + x_2 i_1 + x_3 i_2 + \cdots + x_n i_{n-1}$ is called as dual quaternion form of $X \in R^n$.

Lemma 10.1. Dual quaternion operator $Q = 1 + \upsilon_1 i_1 + \upsilon_2 i_2 + \cdots + \upsilon_{n-1} i_{n-1}$ is a Galilean transformation.

Proof. Since $Q = 1 + \upsilon_1 i_1 + \upsilon_2 i_2 + \cdots + \upsilon_{n-1} i_{n-1}$, we have

$$\begin{aligned} QX &= (1 + \upsilon_1 i_1 + \upsilon_2 i_2 + \cdots + \upsilon_{n-1} i_{n-1})(x_1 + x_2 i_1 + x_3 i_2 + \cdots + x_n i_{n-1}) \\ &= x_1 + (\upsilon_1 x_1 + x_2) i_1 + (\upsilon_2 x_1 + x_3) i_2 + \cdots (\upsilon_{n-1} x_1 + x_n) i_{n-1} \end{aligned}$$

and

$$\|QX\| = \|X\|.$$

So, the dual quaternion operator Q is a Galilean transformation. For any element from Lie algebra , $a \in g(n)$, $a = a_1 i_1 + a_2 i_2 + \cdots + a_{n-1} i_{n-1} \in g(n)$ by using exponential map, we have

$$\begin{aligned} e : g(n) &\to Gal(n) \\ a \to e^a &= e^{a_1 i_1 + a_2 i_2 + \cdots + a_{n-1} i_{n-1}} \\ &= 1 + a_1 i_1 + a_2 i_2 + \cdots + a_{n-1} i_{n-1}. \end{aligned}$$

Corollary 10.1. The transformation Q is a dual quaternion operator. So, dual quaternion operator Q is a Galilean transformation.

Corollary 10.2. Let $a, b \in g(n)$, then $Q(a) = e^a \in Gal(n)$ and $Q(b) = e^b \in Gal(n)$.

Furthermore $Q(a)Q(b) = e^a e^b = e^{a+b} = Q(a+b)$. This implies the addition theorem for the velocity on a Galilean transformation.

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On fuzzy number valued Lebesgue outer measure

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Abstract This paper introduces the concept of fuzzy number valued Lebesgue outer measure. A non-negative countably subadditive function m^* on the power set of a set X by means of a given additive function on an algebra of subsets of X, and a new collection of measurable sets E are constructed , where E satisfy the relation $m^*(A) = m^*(A \cap E) + m^*(A \cap (X \backslash E)\ )$, for any subset A of X .

Keywords Fuzzy number, fuzzy outer measure.

§1. Introduction

There are articles in the literature associated with fuzzy outer measure [3]. We construct fuzzy number valued outer measures and measurable sets that is similar to that of Carathrodory construction. In section 3 we deal with fuzzy number valued Lebesgue outer measure in the real line R and in section 4, the results obtained in section 3 are carried to arbitrary fuzzy number valued measure space (X, Ω, m).

In section 2 we give preliminary ideas relevant to fuzzy numbers.

§2. Basic Definitions

Definition 2.1. Let $F = \{\tilde{n} \backslash \tilde{n} : R \to [0, 1]\}$. For every $\tilde{n} \in F$, $\tilde{n}$ is called a fuzzy number if it satisfies the following properties:

a) $\tilde{n}$ is normal i.e there exists an $x \in R$ such that $\tilde{n}(x) = 1$

b) whenever $\lambda \in [0, 1]$, the $\lambda-$cut , $n_\lambda = \{x : \tilde{n} \geq \lambda\}$ is a closed interval denoted by $[n_\lambda^-, n_\lambda^+]$ We denote the set of all fuzzy numbers on R by F^*.

Remark 2.2. By decomposition theorem of fuzzy sets $\tilde{n} = \bigcup\limits_{\lambda \in [0,1]} \lambda[n_\lambda^-, n_\lambda^+]$.

Remark 2.3. The fuzzy number $\overline{[a, b]}$ is defined by $\overline{[a, b]}(x) = 1$, iff $x \in [a, b]$ and $\overline{[a, b]}(x) = 0$, iff $x \notin [a, b]$. Similarly we can define $\overline{(a, b)}$.

If $a = b$ then $\overline{[a, b]}$ is simply denoted by $\dot{a}$. i.e $\dot{a}$ is defined as $\dot{a}(x) = 1$, iff $x = a$ and $\dot{a}(x) = 0$ iff $x \neq a$. Obviously then $\overline{[a, b]}, \dot{a} \in F^*$.

Further $\overline{[a,b]} = \bigcup_{\lambda\in[0,1]} \lambda[a,b]$. and $\dot{a} = \bigcup_{\lambda\in[0,1]} \lambda[a,a]$.

Remark 2.4. A fuzzy number $\tilde{n}$ which is increasing in the interval (a,b), $\tilde{n}([b,c]) = 1$ and is decreasing in the interval (b,c) is simply denoted by (a,b,c,d). Here $\tilde{n}([b,c]) = 1$ means $\tilde{n}(x) = 1$ for every $x \in [b,c]$.

The fuzzy number $(a,b,c,c+\epsilon)$ where $\epsilon > 0$ is denoted by $(a,\overline{[b,c]})$ and the fuzzy number $(a-\epsilon,a,b,c)$ where $\epsilon > 0$ is denoted by $(\overline{[a,b]},c)$.

Similarly the fuzzy number $(a,b,b,b+\epsilon)$ where $\epsilon > 0$ is denoted by $(a,\dot{b})$ and the fuzzy number $(a-\epsilon,a,a,b)$ where $\epsilon > 0$ is denoted by $(\dot{a},b)$.

Definition 2.5. For every $\tilde{a},\tilde{b} \in F^*$ the sum $\tilde{a}+\tilde{b}$ is defined as $\tilde{c}$ where $c_\lambda^- = a_\lambda^- + b_\lambda^-$ and $c_\lambda^+ = a_\lambda^+ + b_\lambda^+$ for every $\lambda \in (0,1]$.

Definition 2.6. For every $\tilde{a},\tilde{b} \in F^*$ we write$\tilde{a} \leq \tilde{b}$ if $a_\lambda^- \leq b_\lambda^-$ and $a_\lambda^+ \leq b_\lambda^+$ for every $\lambda \in (0,1]$.

§3. Fuzzy number valued Lebesgue outer measure

Definition 3.1. Let $\mu : R \to [0,1]$ be a fuzzy subset of the real line. The fuzzy number valued Lebesgue outer measure for the fuzzy subset μ is defined as $m^*(\mu) = (0,\dot{K})$ where $K = inf \sum_{n=1}^{\infty} \left(\sup_{x\in I_n} \mu(x)\right) l(I_n)$ and the infimum is taken over all countable collection (I_n) of open intervals covering R.

Proposition 3.2. If $\mu_1 \leq \mu_2$, then $m^*(\mu_1) \leq m^*(\mu_2)$.

Proof. $m^*(\mu_1) = (0,\dot{K_1})$ where $K_1 = inf \sum_{n=1}^{\infty} \left(\sup_{x\in I_n} \mu_1(x)\right) l(I_n) \leq (0,\dot{K_2})$ where

$$K_2 = inf \sum_{n=1}^{\infty} \left(\sup_{x\in I_n} \mu_2(x)\right) l(I_n) = m^*(\mu_2).$$

Lemma 3.3. If μ_1 and μ_2 are any two fuzzy sets and $\mu = \mu_1 \vee \mu_2$, then

$$m^*(\mu) \leq m^*(\mu_1) + m^*(\mu_2).$$

Proof. If $m^*(\mu_1) = (0,\dot{\infty})$ or $m^*(\mu_2) = (0,\dot{\infty})$ then the lemma is trivial.

Suppose that $m^*(\mu_1) \neq (0,\dot{\infty})$ and $m^*(\mu_2) \neq (0,\dot{\infty})$. Let $K_1 = \inf \sum_{n=1}^{\infty} \left(\sup_{x\in I_n'} \mu_1(x)\right) l(I_n')$ and $K_2 = \inf \sum_{n=1}^{\infty} \left(\sup_{x\in I_n''} \mu_2(x)\right) l(I_n'')$

Choosing $\epsilon > 0$ we can find countable covers $\{I_1', I_2', \cdots\}$ and $\{I_1'', I_2'', \cdots\}$ of open intervals for R such that $\sum_{n=1}^{\infty} \left(\sup_{x\in I_n'} \mu_1(x)\right) l(I_n') < K_1 + \epsilon/4$ and $\sum_{n=1}^{\infty} \left(\sup_{x\in I_n''} \mu_2(x)\right) l(I_n'') < K_2 + \epsilon/4$. Set another sequence $\{J_1', J_2', \cdots\}$ of pairwise disjoint open intervals such that $R - \bigcup_{n=1}^{\infty} J_n' = T$ (say) is countable and such that each J_n' is contained in some I_i' and I_j''. Choose a sequence of

open intervals $J_1^{"}, J_2^{"}, \cdots$ such that $\sum\limits_{n=1}^{\infty} l(J_n^{"}) < \epsilon/4$ and $T \subseteq \bigcup\limits_{n=1}^{\infty} J_n^{"}$.
Setting $\{I_1, I_2, \cdots\} = \{J_1^{'}, J_1^{"}, J_2^{'}, J_2^{"} \cdots\}$ we can find $\{I_1, I_2, \cdots\}$ is a cover of R.

Also

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\sup_{x\in I_n} \mu_1(x)\right) l(I_n) \\
&\leq \sum_{n=1}^{\infty}\left(\sup_{x\in J_n^{'}} \mu_1(x)\right) l(J_n^{'}) + \sum_{n=1}^{\infty}\left(\sup_{x\in J_n^{"}} \mu_1(x)\right) l(J_n^{"}) \\
&\leq \sum_{n=1}^{\infty}\left(\sup_{x\in I_n^{'}} \mu_1(x)\right) l(I_n^{'}) + \sum_{n=1}^{\infty} l(J_n^{"}) \\
&\leq K_1 + \epsilon/4 + \epsilon/4 = K_1 + \epsilon/2.
\end{aligned}$$

Similarly we find $\sum\limits_{n=1}^{\infty}\left(\sup\limits_{x\in I_n} \mu_2(x)\right) l(I_n) \leq K_2 + \epsilon/2$.

If $K = \inf \sum\limits_{n=1}^{\infty}\left(\sup\limits_{x\in I_n} \mu(x)\right) l(I_n)$, then

$$\begin{aligned}
&K \leq \sum_{n=1}^{\infty}\left(\sup_{x\in I_n} \mu(x)\right) l(I_n) \\
&\leq \sum_{n=1}^{\infty}\left(\sup_{x\in I_n} \mu_1(x)\right) l(I_n) + \\
&\sum_{n=1}^{\infty}\left(\sup_{x\in I_n} \mu_2(x)\right) l(I_n) \\
&\leq K_1 + \epsilon/2 + K_2 + \epsilon/2
\end{aligned}$$

and hence $K \leq K_1 + K_2$.

Therefore

$$m^*(\mu) = (0, \dot{K}) \leq (0, \dot{\overline{K_1 + K_2}}) = (0, \dot{K_1}) + (0, \dot{K_2}) = m^*(\mu_1) + m^*(\mu_2)$$

Remark 3.4. By using the induction on n the result of above lemma can be extended as $m^*(\mu) \leq m^*(\mu_1) + m^*(\mu_2) + \cdots m^*(\mu_n)$ whenever $\mu = \mu_1 \vee \mu_2 \vee \cdots \vee \mu_n$.

Lemma 3.5. $m^*(\mu) = (0, \dot{K})$ where $K = inf \sum\limits_{n=1}^{\infty}\left(\sup\limits_{x\in A_n} \mu(x)\right) m(A_n)$ and the infimum is taken over all sequences (A_n) of Lebesgue measurable subsets of R such that $R = \bigcup\limits_{n=1}^{\infty} A_n$.

Proof. Let $K = \inf \sum\limits_{n=1}^{\infty}\left(\sup\limits_{x\in I_n} \mu(x)\right) l(I_n)$ and $K^{\ '} = \inf \sum\limits_{n=1}^{\infty}\left(\sup\limits_{x\in A_n} \mu(x)\right) m(A_n)$

We have to prove that $K = K^{\ '}$ for which it is enough to show that $K \leq K^{\ '}$.

Suppose that $K^{\ '} < \infty$. Choosing $\epsilon > 0$, we can find a sequence $\{\epsilon_n\}$ with $\epsilon_n > 0$ for all n such that $\sum\limits_{n=1}^{\infty} \epsilon_n < \epsilon/4$. Accordingly we can have a sequence of Lebesgue measurable subsets of R such that $R = \bigcup\limits_{n=1}^{\infty} A_n$ and such that $\sum\limits_{n=1}^{\infty}\left(\sup\limits_{x\in A_n} \mu(x)\right) m(A_n) < K^{\ '} + \epsilon/4$.

To each n, find a sequence of pairwise disjoint open intervals such that $m\left(\left(\bigcup_{k=1}^{\infty} I_{n_k}\right)\setminus A_n\right) < \epsilon_n$ and such that $\bigcup_{k=1}^{\infty} I_{n_k} \triangle\, A_n = M_n$ (say) is countable.

Finding a sequence $\{J_n\}$ of open intervals such that $\bigcup_{n=1}^{\infty} M_n \subseteq \bigcup_{n=1}^{\infty} J_n$ and such that $\sum_{n=1}^{\infty} l(J_n) < \epsilon/4$.

Clearly $\{I_{n_k}, J_n : k = 1, 2, \cdots, n = 1, 2, \cdots\}$ is a cover for R and we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\sup_{x\in I_{n_k}} \mu(x)\right) l(I_{n_k}) + \sum_{n=1}^{\infty}\left(\sup_{x\in J_n} \mu(x)\right) l(J_n) \\
&\leq \sum_{n=1}^{\infty}\left(\sup_{x\in A_n} \mu(x)\right)(m(A_n) + \epsilon_n) + \sum_{n=1}^{\infty} l(J_n) \\
&\leq K^{'} + \sum_{n=1}^{\infty} \epsilon_n + \epsilon/4 \leq K^{'} + \epsilon
\end{aligned}$$

as $0 \leq \mu(x) \leq 1, m\left(\left(\bigcup_{k=1}^{\infty} I_{n_k}\right)\setminus A_n\right) < \epsilon_n$ and $\{I_{n_k}\}, k = 1, 2\cdots$ are pairwise disjoint for every n.

Therefore $K = K^{'}$. and hence the result follows.

Theorem 3.6. If E is a Lebesgue measurable subset of the real line and μ is a fuzzy subset of R then $m^*(\mu) = m^*(\mu \cap E) + m^*(\mu \cap E^c)$.

Proof.

$$\begin{aligned}
m^*(\mu) \quad &= m^*(\mu \cap (E \cup E^c)) = m^*((\mu \cap E) \cup (\mu \cap E^c)) \\
&\leq m^*(\mu \cap E) + m^*(\mu \cap E^c).
\end{aligned}$$

Suppose that $m^*(\mu) = (0, \dot{K}), m^*(\mu \cap E) = (0, \dot{K_1})$ and $m^*(\mu \cap E^c) = (0, \dot{K_2})$.
If $K = \infty$, the result is trivial.

Suppose that $K \neq \infty$. Choosing $\epsilon > 0$ we can find a sequence $\{A_n\}$ of pairwise disjoint measurable subsets of R such that $\bigcup_{n=1}^{\infty} A_n = R$ and

$$\sum_{n=1}^{\infty}\left(\sup_{x\in A_n} \mu(x)\right) m(A_n) < K + \epsilon.$$

Therefore by previous lemma,

$$\begin{aligned} K_1 + K_2 \quad & \leq \sum_{n=1}^{\infty} \left(\sup_{x \in A_n \cap E} \mu(x) \wedge \chi_E(x) \right) m(A_n \cap E) + \\ & \sum_{n=1}^{\infty} \left(\sup_{x \in A_n \cap E^c} \mu(x) \wedge \chi_E(x) \right) m(A_n \cap E^c) + \\ & \sum_{n=1}^{\infty} \left(\sup_{x \in A_n \cap E} \mu(x) \wedge \chi_{E^c}(x) \right) m(A_n \cap E) + \\ & \sum_{n=1}^{\infty} \left(\sup_{x \in A_n \cap E^c} \mu(x) \wedge \chi_{E^c}(x) \right) m(A_n \cap E^c) \\ & = \sum_{n=1}^{\infty} \left(\sup_{x \in A_n \cap E} \mu(x) \right) m(A_n \cap E) + \\ & \sum_{n=1}^{\infty} \left(\sup_{x \in A_n \cap E^c} \mu(x) \right) m(A_n \cap E^c) \\ & \leq \sum_{n=1}^{\infty} \left(\sup_{x \in A_n} \mu(x) \right) m(A_n) \leq K + \epsilon. \end{aligned}$$

Therefore

$$m^*(\mu \cap E) + m^*(\mu \cap E^c) = (0, \dot{K_1}) + (0, \dot{K_2}) = (0, \dot{\overline{K_1 + K_2}}) \leq (0, \dot{K}) = m^*(\mu).$$

Hence the result.

Remark 3.7. Using above theorem we can define Lebesgue measurable fuzzy subset λ as follows:

A fuzzy subset λ is Lebesgue measurable iff $m^*(\mu) = m^*(\mu \cap \lambda) + m^*(\mu \cap \lambda^c)$ for some complement λ^c of λ.

§4. General Fuzzy number valued Lebesgue outer measure

We shall assume X as a nonempty set, Ω denotes a $\sigma-$ algebra of subsets of X and m denotes a positive measure on Ω.

Definition 4.1. Let $\mu : X \to [0, 1]$ be a fuzzy subset of X. The fuzzy number valued outer measure for the fuzzy subset μ is defined as $m^*(\mu) = (0, \dot{K})$ where $K = inf \sum_{n=1}^{\infty} \left(\sup_{x \in A_n} \mu(x) \right) m(A_n)$ and the infimum is taken over all countable collection (A_n) in Ω which cover X.

The following result can be proved that is analogous to lemma 3.3.

If $\mu_1, \mu_2, \cdots \mu_n$ are fuzzy subsets of X and if $\mu = \mu_1 \vee \mu_2 \vee \cdots \vee \mu_n$ then $m^*(\mu) \leq m^*(\mu_1) + m^*(\mu_2) + \cdots m^*(\mu_n)$.

Let $\Omega_1 = E \subseteq X : m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$ for every A $\subseteq X$. Then Ω_1 is $\sigma-$ algebra, $\Omega_1 \subseteq \Omega$. If $m(A) = m^*(A)$ for every $A \in \Omega_1$ then m is a positive measure on Ω_1. If m^* is finite($\sigma-$ finite) then m is finite($\sigma-$ finite). We use $m(A)$ instead of $m^*(A)$ whenever $A \in \Omega_1$.

Lemma 4.2. If μ is fuzzy set on X then $m^*(\mu) = (0, \dot{K})$, where
$K = inf \sum_{n=1}^{\infty} \left(\sup_{x \in A_n} \mu(x) \right) m(A_n)$ and the infimum is taken over all sequences (A_n) of sets in Ω_1 such that $X = \bigcup_{n=1}^{\infty} A_n$.

Proof. The proof is similar to that of lemma 3.5 with the following modifications.

To each n, such that $\sup_{x \in A_n} \mu(x) \neq 0$, choose a sequence such that $A_n \subseteq \bigcup_{k=1}^{\infty} A_{n_k}$, $A_{n_k} \in \Omega$, A_{n_k} are pairwise disjoint, and

$$m\left(\left(\bigcup_{k=1}^{\infty} A_{n_k}\right) \backslash A_n\right) < \epsilon_n.$$

Let B be the set obtained from X after removing all A_{n_k}. Then
$\{A_{n_k} : k = 1, 2, \cdots\} \cup \{B\}$ is a cover for X which leads to the conclusion of the as in lemma 3.5.

Theorem 4.3. If $E \in \Omega_1$ and μ is a fuzzy subset of X , then

$$m^*(\mu) = m^*(\mu \cap E) + m^*(\mu \cap E^c)$$

Proof. Analogous to the proof in lemma 3.6

Remark 4.4. Using above theorem if we wish to define measurable fuzzy subset λ of X as those λ which satisfies $m^*(\mu) = m^*(\mu \cap \lambda) + m^*(\mu \cap \lambda^c)$ for some complement λ^c of λ then λ must be Ω_1- measurable.

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A note to Lagrange mean value Theorem[1]

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Abstract In this paper we prove that the "intermediate point" ξ of Lagrange mean value theorem is a function, furthermore, we study its monotonicity, continuity and derivable property. For application, we give an example to show the incorrect for a proof method in course of calculus teaching.

Keywords Lagrange mean value theorem, "intermediate point" ξ , monotonicity, continuity, derivable property

§1. Introduction

Lagrange mean value theorem is one of most important theorems in calculus. It is an important tool to study the property of function. It sets up a "bridge" between function and derivable function. But Lagrange mean value theorem only affirms existence of "intermediate point" ξ , not states its other properties. Recently, some people have studied asymptotic qualities of ξ and obtained good results. In this paper, on the base of summarizing related results, we study the monotonicity, continuity and derivable property of ξ .

§2. Several lemmas

Lemma 1. (Lagrange mean value theorem) Assume that $f(x)$ is continuous on the closed interval $[a, b]$ and derivable on the open interval (a, b). Then for $\forall x \in (a, b]$, there at least exists a point $\xi \in (a, x)$, such that

$$f'(\xi) = \frac{f(x) - f(a)}{x - a}, \tag{1}$$

In reference[1-4], they obtain some important results, among them the most typical result is as follows.

Lemma 2. Assume that $f(x)$ is a first-order continuous and derivable function on interval $[a, b]$, and $f'(x) - f'(a)$ is α-order infinitesimal of $x - a$, where $\alpha > 0$. Then ξ with (1) has asymptotic estimator

$$\lim_{b \to a} \frac{\xi - a}{b - a} = \left(\frac{1}{a+1}\right)^{\frac{1}{a}}. \tag{2}$$

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§3. Main results and their proofs

Assume that $f(x)$ satisfies the conditions of lemma 1 when $x \in [a, b]$, then for $\forall x \in (a, b]$, when a is fixed, the "intermediate point" ξ changes with x and ξ has important properties as follows.

Theorem. Assume that $f(x)$ is continuous on the closed interval $[a, b]$ and derivable on the open interval (a, b), and $f(x)$ is second-order continuous and derivable in interval (a, b), $f'(x)$ is strictly monotone in interval(a, b) , $f''(x)$ is always positive or always negative in interval (a, b), then we have

(i) The point ξ with (1) is a uniform function about x, notes $\xi = \xi(x)$;

(ii) $\xi = \xi(x)$ is a monotone increasing function about x;

(iii) $\xi = \xi(x)$ is a continuous function about x;

(iv) $\xi = \xi(x)$ is a derivable function about x and

$$\xi'(x) = \frac{f'(x) - f'(\xi(x))}{(x-a)f''(\xi(x))}. \tag{3}$$

Proof. (i) Because $f'(x)$ is strictly monotone in interval (a, b), we can easily prove that (i) holds.

(ii) Assume that $f'(x)$ is monotone increasing in interval (a, b), for $\forall x_1, x_2 \in (a, b)$ and $x_1 < x_2$, by given condition and lemma 1 we have

$f(x_2) - f(a) = f'(\xi(x_2))(x_2 - a)$, $f(x_1) - f(a) = f'(\xi(x_1))(x_1 - a)$.

So $f(x_2) - f(x_1) = [f'(\xi(x_2)) - f'(\xi(x_1))](x_1 - a) + f'(\xi(x_2))(x_2 - x_1)$. Also by $f(x_2) - f(x_1) = f'(\eta)(x_2 - x_1)$, hence

$[f'(\eta) - f'(\xi(x_2))](x_2 - x_1) = [f'(\xi(x_2)) - f'(\xi(x_1))](x_1 - a)$,

where $x_1 < \eta < x_2,\ a < \xi(x_1) < x_1,\ a < \xi(x_2) < x_2,\ a < x_1 < x_2 < b$.Because $f'(x)$ is monotone increasing, we have $f'(\eta) > f'(\xi(x_1))$, and

$$\begin{aligned} f'(\eta) - f'(\xi(x_2))(x_2 - a) &= f'(\eta)(x_2 - x_1) + f'(\eta)(x_1 - a) - f'(\xi(x_2))(x_2 - a) \\ &> f'(\eta)(x_2 - x_1) + f'(\xi(x_1))(x_1 - a) - f'(\xi(x_2))(x_2 - a). \end{aligned}$$

But $f'(\eta)(x_2 - x_1) = f(x_2) - f(x_1)$,

$f'(\xi(x_1))(x_1 - a) = f(x_1) - f(a)$,

$f'(\xi(x_2))(x_2 - a) = f(x_2) - f(a)$,

So $[f'(\eta) - f'(\xi(x_2))](x_2 - a) > 0$,

$f'(\eta) - f'(\xi(x_2)) > 0$,

$f'(\xi(x_2)) - f'(\xi(x_1)) > 0$.

By monotone increasing $f'(x)$, we have

$\xi(x_2) > \xi(x_1)$.

When $f'(x)$ is monotone decreasing in interval (a, b), the proof is same to above proof.

(iii) By given conditions and Lemma 1, we have

$$f'(\xi(x)) = \frac{f(x) - f(a)}{x - a},$$

and

$$f'(\xi(x+h)) = \frac{f(x+h) - f(a)}{x + h - a},$$

so

$$f'(\xi(x+h)) - f'(\xi(x)) = \frac{(x-a)[f(x+h)-f(a)] - h[f(x)-f(a)]}{(x+h-a)(x-a)}.$$

By (ii), we know that $\xi = \xi(x)$ is a monotone function about x. When $h \neq 0$, also by Lemma 1, we have

$$f'(\xi(x+h)) - f'(\xi(x)) = f''(\eta)[\xi(x+h) - \xi(x)],$$

where η lies between $\xi(x+h)$ and $\xi(x)$, so

$$\lim_{h\to 0}[\xi(x+h) - \xi(x)] = \lim_{h\to 0} \frac{(x-a)[f(x+h)-f(a)] - h[f(x)-f(a)]}{f''(\eta)(x+h-a)(x-a)} = 0.$$

That is to say, $\xi = \xi(x)$ is continuous in interval (a, b).

(iv) By definition of derivative, we have

$$\begin{aligned}
\xi'(x) &= \lim_{h\to 0} \frac{\xi(x+h) - \xi(x)}{h} \\
&= \lim_{h\to 0} \frac{(x-a)\dfrac{f(x+h)-f(x)}{h} - [f(x)-f(a)]}{f''(\eta)(x+h-a)(x-a)} \\
&= \frac{(x-a)f'(x) - f(x) + f(a)}{f''(\xi(x))(x-a)^2} \\
&= \frac{f'(x) - f'(\xi(x))}{(x-a)f''(\xi(x))}.
\end{aligned}$$

Thus, the proof is complete.

Finally, it deserves to mention that, in the course of teaching higher mathematics, some people consider the application of this theorem as an incorrect proof method. In fact, this thinking is not right. we give an example.

Example. Assume that $f(x)$ is quadratic differentiable on the closed interval $[a, b]$ and $f''(x) > 0$, we try to prove that the function

$$g(x) = \frac{f(x) - f(a)}{x - a}$$

is a monotone increasing function in the open interval (a, b). Proof By Lemma1, we have

$$f(x) - f(a) = f'(\xi(x))(x-a),\ a < \xi(x) < x \leqslant b.$$

So

$$g(x) = \frac{f(x) - f(a)}{x - a} = f'(\xi(x)).$$

By above theorem we know that $\xi = \xi(x)$ is a monotone increasing and derivable function in the open interval (a, b), also by derivative rules of compound function, we have

$$g'(x) = f''(\xi(x))\xi'(x),$$

but $f'(x) > 0$ and $\xi'(x) > 0$, so

$$g'(x) = f''(\xi(x))\xi'(x) > 0.$$

Thus, the proof is complete.

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Some properties of (α, β)-fuzzy BG-algebras

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Abstract By two relations belonging to $(\in)$ and quasi-coincidence (q) between fuzzy points and fuzzy sets, we define the concept of (α, β)-fuzzy subalgebras where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$. We state and prove some theorems in (α, β)-fuzzy BG-algebras.

Keywords BG-algebra, (α, β)-fuzzy subalgebra, fuzzy point.

§1. Introduction

Y. Imai and K. Iseki [3] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [7] J. Neggers and H. S. Kim introduced the notion of d-algebras, which is generalization of BCK-algebras and investigated relation between d-algebras and BCK-algebras. Also they introduced the notion of B-algebras [6]. In [4] C. B. Kim, H. S. Kim introduced the notion of BG-algebras which is a generalization of B-algebras. In 1980, P. M. Pu and Y. M. Liu [8], introduced the idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is used to generate some different types of fuzzy subgroups, called (α, β)-fuzzy subgroups, introduced by Bhakat and Das [2]. In particular, $\{\in, \in \vee q\}$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. In this note we introduced the notion of (α, β)-fuzzy BG-algebras. We state and prove some theorems discussed in (α, β)-fuzzy BG-subalgebras and level subalgebras.

§2. Preliminary

Definition 2.1. [4] A BG-algebra is a non-empty set X with a consonant 0 and a binary operation $*$ satisfying the following axioms:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $(x * y) * (0 * y) = x$, for all $x, y \in X$.

For brevity we also call X a BG-algebra. In X we can define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$.

Theorem 2.2. [4] In a BG-algebra X, we have the following properties:

Theorem 2.2. [4] In a BG-algebra X, we have the following properties:

For all $x, y \in X$,

(i) $0 * (0 * x) = x$,

(ii) if $x * y = 0$, then $x = y$,

(iii) if $0 * x = 0 * y$, then $x = y$,

(iv) $(x * (0 * x)) * x = x$.

Theorem 2.3. [4] A non-empty subset I of a BG-algebra X is called a subalgebra of X if $x * y \in I$ for any $x, y \in I$.

A mapping $f : X \longrightarrow Y$ of BG-algebras is called a BG-homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

We now review some fuzzy logic concepts (see [2] and [10]).

Let X be a set. A fuzzy set A on X is characterized by a membership function $\mu_A : X \longrightarrow [0, 1]$.

Let $f : X \longrightarrow Y$ be a function and B a fuzzy set of Y with membership function μ_B. The inverse image of B, denoted by $f^{-1}(B)$, is the fuzzy set of X with membership function $\mu_{f^{-1}(B)}$ defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$.

Conversely, let A be a fuzzy set of X with membership function μ_A. Then the image of A, denoted by $f(A)$, is the fuzzy set of Y such that

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

A fuzzy set μ of a set X of the form

$$\mu(y) := \begin{cases} t & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

where $t \in (0, 1]$ is called a fuzzy point with support x and value t and is denoted by x_t.

Consider a fuzzy point x_t, a fuzzy set μ on a set X and $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$, we define $x_t \alpha \mu$ as follow:

(i) $x_t \in \mu$ (resp. $x_t q \mu$) means that $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$) and in this case we said that x_t belong to (resp. quasi-coincident with) fuzzy set μ.

(ii) $x_t \in \vee q \mu$ (resp. $x_t \in \wedge q \mu$) means that $x_t \in \mu$ or $x_t q \mu$ (resp. $x_t \in \mu$ and $x_t q \mu$).

Definition 2.4. [1] Let μ be a fuzzy set of a BG-algebra X. Then μ is called a fuzzy BG-algebra (subalgebra) of X if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in X$.

Example 2.5. [1] Let $X = \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Then $(X,*,0)$ is a BG-algebra. Define a fuzzy set $\mu : X \to [0,1]$ on X, by $\mu(0) = \mu(1) = t_0$ and $\mu(2) = \mu(3) = t_1$, for $t_0, t_1 \in [0,1]$ and $t_0 > t_1$. Then μ is a fuzzy BG-algebra of X.

Definition 2.6. [2] Let μ be a fuzzy set of X. Then the upper level set $U(\mu;\lambda)$ of X is defined as following :

$$U(\mu;\lambda) = \{x \in X \mid \mu(x) \geq \lambda\}.$$

Definition 2.7. Let $f : X \longrightarrow Y$ be a function. A fuzzy set μ of X is said to be f-invariant, if $f(x) = f(y)$ implies that $\mu(x) = \mu(y)$, for all $x, y \in X$.

§3. (α,β)-fuzzy BG-algebras

From now on X is a BG-algebra and α, $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$ unless otherwise specified. By $x_t\overline{\alpha}\mu$ we mean that $x_t\alpha\mu$ does not hold.

Theorem 3.1. Let μ be a fuzzy set of X. Then μ is a fuzzy BG-algebra if and only if

$$x_{t_1}, y_{t_2} \in \mu \Rightarrow (x*y)_{\min(t_1,t_2)} \in \mu, \tag{1}$$

for all $x, y \in X$ and $t_1, t_2 \in [0,1]$.

Proof. Assume that μ is a fuzzy BG-algebra. Let $x, y \in X$ and $x_{t_1}, y_{t_2} \in \mu$, for $t_1, t_2 \in [0,1]$. Then $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$, by hypothesis we can conclude that

$$\mu(x*y) \geq \min(\mu(x), \mu(y)) \geq \min(t_1, t_2).$$

Hence $(x*y)_{\min(t_1,t_2)} \in \mu$.

Conversely, Since $x_{\mu(x)} \in \mu$ and $y_{\mu(y)} \in \mu$ for all $x, y \in X$, then $(x*y)_{\min(\mu(x),\mu(y))} \in \mu$. Therefore $\mu(x*y) \geq \min(\mu(x), \mu(y))$.

Note that if μ is a fuzzy set of X defined by $\mu(x) \leq 0.5$ for all $x \in X$, then the set $\{x_t \mid x_t \in \wedge q\mu\}$ is empty.

Definition 3.2. A fuzzy set μ of X is said to be an (α,β)-fuzzy subalgebra of X, where $\alpha \neq \in \wedge q$, if it satisfies the following condition:

$$x_{t_1}\alpha\mu, y_{t_2}\alpha\mu \Rightarrow (x*y)_{\min(t_1,t_2)}\beta\mu$$

for all t_1, $t_2 \in (0,1]$.

Proposition 3.3. μ is an $(\in,\in)$-fuzzy subalgebra of X if and only if for all $t \in [0,1]$, the nonempty level set $U(\mu;t)$ is a subalgebra of X.

Proof. The proof follows from Theorem 3.1.

Example 3.4. Let $X = \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X, *, 0)$ is a BG-algebra. Let μ be a fuzzy set in X defined $\mu(0) = 0.6$, $\mu(1) = 0.7$ and $\mu(2) = \mu(3) = 0.3$. Then μ is an $(\in, \in \vee q)$-fuzzy subalgebra of X. But

(1) μ is not an $(\in, \in)$-fuzzy subalgebra of X since $1_{0.62} \in \mu$ and $1_{0.66} \in \mu$, but $(1 * 1)_{\min(0.62,0.66)} = 0_{0.62} \overline{\in} \mu$.

(2) μ is not a $(q, \in \vee q)$-fuzzy subalgebra of X since $1_{0.41} q \mu$ and $2_{0.77} q \mu$, but $(1*2)_{\min(0.41,0.77)} = 3_{0.41} \overline{\in \vee q} \mu$.

(3) μ is not an $(\in \vee q, \in \vee q)$-fuzzy subalgebra of X since $1_{0.5} \in \vee q \mu$ and $3_{0.8} \in \vee q \mu$, but $(1 * 3)_{\min(0.5,0.8)} = 2_{0.5} \overline{\in \vee q} \mu$.

Theorem 3.5. Let μ be a fuzzy set. Then the following diagram shows the relationship between (α, β)-fuzzy subalgebras of X, where α, β are one of $\in$ and q.

$$\begin{tikzcd}
& (\alpha, \alpha \vee \beta) & \\
(\alpha, \beta) \arrow[ur] & & (\alpha, \alpha) \arrow[ul] \\
& (\alpha, \alpha \wedge \beta) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

and also we have

$$\begin{tikzcd}
& (\in \vee q, \in \vee q) & \\
(\in \vee q, q) \arrow[ur] & & (\in \vee q, \in) \arrow[ul] \\
& (\in \vee q, \in \wedge q) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Proof. The proof is easy.

Proposition 3.6. If μ is a nonzero (α, β)-fuzzy subalgebra of X, then $\mu(0) > 0$.

Proof. Assume that $\mu(0) = 0$. Since μ is non-zero, then there exists $x \in X$ such that

$\mu(x) = t > 0$. Thus $x_t\alpha\mu$ for $\alpha = \in$ or $\alpha = \in \vee q$, but $(x*x)_{\min(t,t)} = 0_t\overline{\beta}\mu$. This is a contradiction. Also $x_1\alpha\mu$ where $\alpha = q$, since $\mu(x)+1 = t+1 > 1$. But $(x*x)_{\min(1,1)} = 0_1\overline{\beta}\mu$, which is a contradiction. Hence $\mu(0) > 0$.

For a fuzzy set μ in X, we denote the support μ by, $X_0 := \{x \in X \mid \mu(x) > 0\}$.

Proposition 3.7. If μ is a nonzero $(\in, \in\vee q)$-fuzzy subalgebra of X, then the set X_0 is a subalgebra of X.

Proof. Let $x, y \in X_0$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Suppose that $\mu(x*y) = 0$, then $x_{\mu(x)} \in \mu$ and $y_{\mu(y)} \in \mu$, but $\mu(x*y) = 0 < \min(\mu(x), \mu(y))$ and $\mu(x*y) + \min(\mu(x), \mu(y)) \leq 1$, i.e $(x*y)_{\min(\mu(x),\mu(y))}\overline{\in\vee q}\mu$, which is a contradiction . Hence $x*y \in X_0$. Therefore X_0 is a subalgebra of X.

Proposition 3.8. If μ is a nonzero $(q, \in\vee q)$-fuzzy subalgebra of X, then the set X_0 is a subalgebra of X.

Proof. Let $x, y \in X_0$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Thus $\mu(x)+1 > 1$ and $\mu(y)+1 > 1$ imply that $x_1q\mu$ and $y_1q\mu$. If $\mu(x*y) = 0$, then $\mu(x*y) < 1 = \min(1,1)$ and $\mu(x*y)+\min(1,1) \leq 1$. Thus $(x*y)_{\min(1,1)}\overline{\in\vee q}\mu$, which is a contradiction. It follows that $\mu(x*y) > 0$ and so $x*y \in X_0$.

Theorem 3.9. Let μ be a nonempty (α,β)-fuzzy subalgebra, where $\alpha,\beta \in \{\in, q, \in\vee q, \in\wedge q\}$ and $\alpha \neq \in\wedge q$. Then X_0 is a subalgebra of X.

Proof. The proof follows from Theorem 3.5 and Propositions 3.7 and 3.8.

Theorem 3.10. Any non-zero (q, q)-fuzzy subalgebra of X is constant on X_0.

Proof. Let μ be a non-zero (q, q)-fuzzy subalgebra of X. On the contrary, assume that μ is not constant on X_0. Then there exists $y \in X_0$ such that $t_y = \mu(y) \neq \mu(0) = t_0$. Suppose that $t_y < t_0$ and so $1-t_0 < 1-t_y < 1$. Thus there exists $t_1, t_2 \in (0,1)$ such that $1-t_0 < t_1 < 1-t_y < t_2 < 1$. Then $\mu(0)+t_1 = t_0+t_1 > 1$ and $\mu(y)+t_2 = t_y+t_2 > 1$. So $0_{t_1}q\mu$ and $y_{t_2}q\mu$. Since

$$\mu(y*0) + \min(t_1,t_2) = \mu(y) + t_1 = t_y + t_1 < 1,$$

we get that $(y*0)_{\min(t_1,t_2)}\overline{q}\mu$, which is a contradiction. Now let $t_y > t_0$ and $t_0 \neq 1$. Then $\mu(y) + (1-t_0) = t_y + 1 - t_0 > 1$, i.e $y_{1-t_0}q\mu$. Since

$$\mu(y*y) + (1-t_0) = \mu(0) + 1 - t_0 = t_0 + 1 - t_0 = 1,$$

then we get that $(y*y)_{\min(1-t_0,1-t_0)}\overline{q}\mu$, which is a contradiction. Therefore μ is constant on X_0.

Theorem 3.11. μ is a non-zero (q, q)-fuzzy subalgebra if and only if there exists subalgebra S of X such that

$$\mu(x) = \begin{cases} t & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

for some $t \in (0,1]$.

Proof. Let μ be a non-zero (q, q)-fuzzy subalgebra. Then by Proposition 3.6 and Theorems

3.9 and 3.10 we have $\mu(0) > 0$, X_0 is a subalgebra of X and

$$\mu(x) = \begin{cases} \mu(0) & \text{if } x \in X_0 \\ 0 & \text{otherwise} \end{cases}$$

Conversely, let $x_{t_1} q \mu$ and $y_{t_2} q \mu$, for $t_1, t_2 \in (0, 1]$. Then $\mu(x) + t_1 > 1$ and $\mu(y) + t_2 > 1$ imply that $\mu(x) \neq 0$ and $\mu(y) \neq 0$. Thus $x, y \in S$ and so $x * y \in S$. Hence $\mu(x * y) + \min(t_1, t_2) = t + \min(t_1, t_2) > 1$. Therefore μ is a (q, q)-fuzzy subalgebra of X.

Theorem 3.12. μ is a non-zero (q, q)-fuzzy subalgebra of X if and only if $U(\mu; \mu(0)) = X_0$ and for all $t \in [0, 1]$, the nonempty level set $U(\mu; t)$ is a fuzzy subalgebra of X.

Proof. Let μ be a non-zero (q, q)-fuzzy subalgebra. Then by Theorem 3.11 we have

$$\mu(x) = \begin{cases} \mu(0) & \text{if } x \in X_0 \\ 0 & \text{otherwise} \end{cases}$$

So it is easy to check that $U(\mu; \mu(0)) = X_0$. Let $x, y \in U(\mu; t)$, for $t \in [0, 1]$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. If $t = 0$, then it is clear that $x * y \in U(\mu; 0)$. Now let $t \in (0, 1]$. Then $x, y \in X_0$ and so $x * y \in X_0$. Hence $\mu(x * y) = \mu(0) \geq t$. Therefore $U(\mu; t)$ is a subalgebra of X.
Conversely, since $U(\mu; \mu(0)) = X_0$ and $0 \in U(\mu; \mu(0))$, then X_0 is a subalgebra of X. Also $U(\mu; \mu(0)) = X_0$ and $X \neq \emptyset$ imply that μ is non-zero. Now let $x \in X_0$. Then $\mu(x) \geq \mu(0)$ and $\mu(x) > 0$. Since $U(\mu; \mu(x)) \neq \emptyset$, so $U(\mu; \mu(x))$ is a subalgebra of X. Then $0 \in U(\mu; \mu(x))$ imply that $\mu(0) \geq \mu(x)$. Hence $\mu(x) = \mu(0)$, for all $x \in X_0$ i.e

$$\mu(x) = \begin{cases} \mu(0) & \text{if } x \in X_0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore by Theorem 3.11 μ is a (q, q)-fuzzy subalgebra of X.

Example 3.13. Let $X = \{0, 1, 2, 3\}$ be BG-algebra in Example 3.3. Define fuzzy set μ on X by

$$\mu(0) = 0.6, \quad \mu(1) = \mu(2) = \mu(3) = 0.3.$$

Then $X_0 = X$, $U(\mu; \mu(0)) = \{0\} \neq X_0$ and also

$$U(\mu; t) = \begin{cases} X & \text{if } 0 \leq t \leq 0.3 \\ \{0\} & \text{if } 0.3 < t \leq 0.6 \\ \emptyset & \text{if } t > 0.6 \end{cases}$$

is a subalgebra of X, while by Theorem 3.11, μ is not a (q, q)-fuzzy subalgebra.

Theorem 3.14. Every (q, q)-fuzzy subalgebra is an $(\in, \in)$-fuzzy subalgebra.

Proof. The proof follows from Theorem 3.12 and Proposition 3.3.

Note that in Example 3.13 μ is an $(\in, \in)$-fuzzy subalgebra, while it is not a (q, q)-fuzzy subalgebra. So the converse of the above theorem is not true in general.

Theorem 3.15. If μ is a non-zero fuzzy set of X. Then there exists subalgebra S of X such that $\mu = \chi_S$ if and only if μ is an (α, β)-fuzzy subalgebra of X, where (α, β) is one of the following forms:

(i) $(\in, q)$, (ii) $(\in, \in \wedge q)$,

(iii) $(q, \in)$, (iv) $(q, \in \wedge q)$,

(v) $(\in \vee q, q)$, (vi) $(\in \vee q, \in \wedge q)$,

(vii) $(\in \vee q, \in)$.

Proof. Let $\mu = \chi_S$. We show that μ is $(\in, \in \wedge q)$-fuzzy subalgebra. Let $x_{t_1} \in \mu$ and $x_{t_2} \in \mu$, for $t_1, t_2 \in (0, 1]$. Then $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$ imply that $x, y \in S$. Thus $x * y \in S$, i.e $\mu(x * y) = 1$. Therefore $\mu(x * y) \geq \min(t_1, t_2)$ and $\mu(x * y) + \min(t_1, t_2) > 1$, i.e $(x * y)_{\min(t_1,t_2)} \in \wedge q\mu$. Similar to above argument, we can see that μ is an (α, β)-fuzzy subalgebra of X, where (α, β) is one of the above forms.
Conversely, we show that $\mu = \chi_{X_0}$. Suppose that there exists $x \in X_0$ such that $\mu(x) < 1$. Let $\alpha =\in$, choose $t \in (0, 1]$ such that $t < \min(1 - \mu(x), \mu(x), \mu(0))$. Then $x_t\alpha\mu$ and $0_t\alpha\mu$, but $(x * 0)_{\min(t,t)} = x_t\overline{\beta}\mu$, where $\beta = q$ or $\beta =\in \wedge q$. Which is a contradiction. If $\alpha = q$, then $x_1\alpha\mu$ and $0_1\alpha\mu$, while $(x * 0)_{\min(1,1)} = x_1\overline{\beta}\mu$ where $\beta =\in$ or $\beta =\in \wedge q$, which is a contradiction. Now let $\alpha =\in \vee q$ and choose $t \in (0, 1]$ such that $x_t \in \mu$ but $x_t\overline{q}\mu$. Then $x_t\alpha\mu$ and $0_1\alpha\mu$ but $(x * 0)_{\min(t,1)} = x_t\overline{\beta}\mu$ for $\beta = q$ or $\beta =\in \wedge q$, which is a contradiction. Finally we have $x_1 \in \vee q\mu$ and $0_1 \in \vee q\mu$ but $(x * 0)_{\min(1,1)} = x_1\overline{\in}\mu$, which is a contradiction. Therefore $\mu = \chi_{X_0}$.

Theorem 3.16. Let S be a subalgebra of X and let μ be a fuzzy set of X such that

(a) $\mu(x) = 0$ for all $x \in X \backslash S$,

(b) $\mu(x) \geq 0.5$ for all $x \in S$.

Then μ is a $(q, \in \vee q)$-fuzzy subalgebra of X.

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1}q\mu$ and $y_{t_2}q\mu$. Then we get that $\mu(x) + t_1 > 1$ and $\mu(y) + t_2 > 1$. We can conclude that $x * y \in S$, since in otherwise $x \in X \backslash S$ or $y \in X \backslash S$ and therefore $t_1 > 1$ or $t_2 > 1$ which is a contradiction. If $\min(t_1, t_2) > 0.5$, then $\mu(x*y)+\min(t_1, t_2) > 1$ and so $(x*y)_{\min(t_1,t_2)}q\mu$. If $\min(t_1, t_2) \leq 0.5$, then $\mu(x*y) \geq \min(t_1, t_2)$ and thus $(x * y)_{\min(t_1,t_2)} \in \mu$. Hence $(x * y)_{\min(t_1,t_2)} \in \vee q\mu$.

Theorem 3.17. Let μ be a $(q, \in \vee q)$-fuzzy subalgebra of X such that μ is not constant on the set X_0. Then there exists $x \in X$ such that $\mu(x) \geq 0.5$. Moreover, $\mu(x) \geq 0.5$ for all $x \in X_0$.

Proof. Assume that $\mu(x) < 0.5$ for all $x \in X$. Since μ is not constant on X_0, then there exists $x \in X_0$ such that $t_x = \mu(x) \neq \mu(0) = t_0$. Let $t_0 < t_x$. Choose $\delta > 0.5$ such that $t_0 + \delta < 1 < t_x + \delta$. It follows that $x_\delta q\mu$, $\mu(x * x) = \mu(0) = t_0 < \delta = \min(\delta, \delta)$ and $\mu(x*x)+\min(\delta, \delta) = \mu(0)+\delta = t_0+\delta < 1$. Thus $(x*x)_{\min(\delta,\delta)}\overline{\in \vee q}\mu$, which is a contradiction. Now, if $t_x < t_0$ then we can choose $\delta > 0.5$ such that $t_x + \delta < 1 < t_0 + \delta$. Thus $0_\delta q\mu$ and $x_1q\mu$, but $(x * 0)_{\min(1,\delta)} = x_\delta\overline{\in \vee q}\mu$, because $\mu(x) < 0.5 < \delta$ and $\mu(x) + \delta = t_x + \delta < 1$, which is a contradiction. Hence $\mu(x) \geq 0.5$ for some $x \in X$. Now we show that $\mu(0) \geq 0.5$. On the contrary, assume that $\mu(0) = t_0 < 0.5$. Since there exists $x \in X$ such that $\mu(x) = t_x \geq 0.5$, it follows that $t_0 < t_x$. Choose $t_1 > t_0$ such that $t_0+t_1 < 1 < t_x+t_1$. Then $\mu(x)+t_1 = t_x+t_1 > 1$, and so $x_tq\mu$. Thus we can conclude that

$$\mu(x * x) + \min(t_1, t_1) = \mu(0) + t_1 = t_0 + t_1 < 1,$$

and

$$\mu(x * x) = \mu(0) = t_0 < t_1 = \min(t_1, t_1).$$

Therefore $(x * x)_{\min(t_1,t_1)} \overline{\in \vee q} \mu$, which is a contradiction. Thus $\mu(0) \geq 0.5$. Finally we prove that $\mu(x) \geq 0.5$ for all $x \in X_0$. On the contrary, let $x \in X_0$ and $t_x = \mu(x) < 0.5$. Consider $0 < t < 0.5$ such that $t_x + t < 0.5$. Then $\mu(x) + 1 = t_x + 1 > 1$ and $\mu(0) + (0.5 + t) > 1$, imply that $x_1 q \mu$ and $0_{0.5+t} q \mu$. But $(x * 0)_{\min(1,0.5+t)} = x_{0.5+t} \overline{\in \vee q} \mu$, since $\mu(x * 0) = \mu(x) < 0.5 + t$ and $\mu(x) + 0.5 + t = t_x + 0.5 + t < 0.5 + 0.5 = 1$. Which is a contradiction. Therefore $\mu(x) \geq 0.5$ for all $x \in X_0$.

Theorem 3.18. Let μ be a non-zero fuzzy set of X. Then μ is a $(q, \in \vee q)$-fuzzy subalgebra of X if and only if there exists subalgebra S of X such that

$$\mu(x) = \begin{cases} a & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases} \quad or \quad \mu(x) = \begin{cases} \geq 0.5 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

for some $a \in (0, 1]$.

Proof. Let μ be a $(q, \in \vee q)$-fuzzy subalgebra of X. If μ is constant on X_0, then $\mu(x) = \begin{cases} \mu(0) & \text{if } x \in X_0 \\ 0 & \text{otherwise} \end{cases}$. If μ is not constant on X_0, then by Theorem 3.17 we have $\mu(x) = \begin{cases} \geq 0.5 & \text{if } x \in X_0 \\ 0 & \text{otherwise} \end{cases}$. Conversely, the proof follows from Theorems 3.11, 3.5 and 3.16.

Theorem 3.19. Let μ be a non-zero $(q, \in \vee q)$-fuzzy subalgebra of X. Then the nonempty level set $U(\mu; t)$ is a subalgebra of X, for all $t \in [0, 0.5]$.

Proof. If μ is constant on X_0, then by Theorem 3.11, μ is a (q, q)-fuzzy subalgebra. Thus by Theorem 3.12 we have the nonempty level set $U(\mu; t)$ is a subalgebra of X, for $t \in [0, 1]$. If μ is not constant on X_0, then by Theorem 3.17, we have $\mu(x) = \begin{cases} \geq 0.5 & \text{if } x \in X_0 \\ 0 & \text{otherwise} \end{cases}$. Now we show that the nonempty level set $U(\mu; t)$ is a subalgebra of X for $t \in [0, 0.5]$. If $t = 0$, then it is clear that $U(\mu; t)$ is a subalgebra of X. Now let $t \in (0, 0.5]$ and $x, y \in U(\mu; t)$. Then $\mu(x), \mu(y) \geq t > 0$ imply that $x, y \in X_0$. Thus $x * y \in X_0$ and so $\mu(x * y) \geq 0.5 \geq t$. Therefore $x * y \in U(\mu; t)$.

Theorem 3.20. Let μ be a non-zero fuzzy set of X, $U(\mu; 0.5) = X_0$ and the nonempty level set $U(\mu; t)$ is a subalgebra of X, for all $t \in [0, 1]$. Then μ is a $(q, \in \vee q)$-fuzzy subalgebra of X.

Proof. Since $\mu \neq 0$ we get that $X_0 \neq \emptyset$. Thus by hypothesis we have $U(\mu; 0.5) \neq \emptyset$ and so X_0 is a subalgebra of X. Also $\mu(x) \geq 0.5$, for all $x \in X_0$ and $\mu(x) = 0$, if $x \notin X_0$. Therefore by Theorem 3.16, μ is a $(q, \in \vee q)$-fuzzy subalgebra of X.

Theorem 3.21. A fuzzy set μ of X is an $(\in, \in \vee q)$-fuzzy subalgebra of X if and only if $\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5)$, for all $x, y \in X$.

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy subalgebra of X and $x, y \in X$. If $\mu(x)$ or $\mu(y) = 0$, then $\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5)$. Now let $\mu(x)$ and $\mu(y) \neq 0$. If $\min(\mu(x), \mu(y)) < 0.5$, then $\mu(x * y) \geq \min(\mu(x), \mu(y))$. Since, assume that $\mu(x * y) < \min(\mu(x), \mu(y))$, then there exists

$t > 0$ such that $\mu(x * y) < t < \min(\mu(x), \mu(y))$. Thus $x_t \in \mu$ and $y_t \in \mu$ but $(x * y)_{\min(t,t)} = (x*y)_t \overline{\in \vee q} \mu$, since $\mu(x*y) < t$ and $\mu(x*y)+t < 1 < 2t < 1$, which is a contradiction. Hence if $\min(\mu(x), \mu(y)) < 0.5$, then $\mu(x * y) \geq \min(\mu(x), \mu(y))$. If $\min(\mu(x), \mu(y)) \geq 0.5$, then $x_{0.5} \in \mu$ and $y_{0.5} \in \mu$. So we can get that

$$(x * y)_{\min(0.5,0.5)} = (x * y)_{0.5} \in \vee q\mu.$$

Then $\mu(x * y) > 0.5$. Consequently, $\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5)$, for all $x, y \in X$.

Conversely, let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. So $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$. Then by hypothesis we have $\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5) \geq \min(t_1, t_2, 0.5)$. If $\min(t_1, t_2) \leq 0.5$, then $\mu(x * y) \geq \min(\mu(x), \mu(y))$. If $\min(t_1, t_2) > 0.5$, then $\mu(x * y) \geq 0.5$. Thus $\mu(x * y) + \min(t_1, t_2) > 1$. Therefore $(x * y)_{\min(t_1,t_2)} \in \vee q\mu$.

Theorem 3.22. Let μ be an $(\in, \in \vee q)$-fuzzy subalgebra of X.

(i) If there exists $x \in X$ such that $\mu(x) \geq 0.5$, then $\mu(0) \geq 0.5$;

(ii) If $\mu(0) < 0.5$, then μ is an $(\in, \in)$-fuzzy subalgebra of X.

Proof. (i) Let $\mu(x) \geq 0.5$. Then by hypothesis we have $\mu(0) = \mu(x*x) \geq \min(\mu(x), \mu(x), 0.5) = 0.5$.

(ii) Let $\mu(0) < 0.5$. Then by (i) $\mu(x) < 0.5$, for all $x \in X$. Now let $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$, for $t_1, t_2 \in (0, 1]$. Then $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$. Thus $\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5) \geq \min(t_1, t_2, 0.5) = \min(t_1, t_2)$. Therefore $(x * y)_{\min(t_1,t_2)} \in \mu$.

Lemma 3.23. Let μ be a non-zero $(\in, \in \vee q)$ fuzzy subalgebra of X. Let $x, y \in X$ such that $\mu(x) < \mu(y)$. Then

$$\mu(x * y) = \begin{cases} \mu(x) & \text{if } \ \mu(y) < 0.5 \ \ or \ \ \mu(x) < 0.5 \leq \mu(y) \\ \geq 0.5 & \text{if } \ \ \mu(x) \geq 0.5 \end{cases} .$$

Proof. Let $\mu(y) < 0.5$. Then we have $\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5) = \mu(x)$. Also

$$\mu(x) = \mu((x * y) * (0 * y)) \geq \min\{\mu(x * y), \mu(0 * y), 0.5\} \qquad (1)$$

Now we show that $\mu(0*y) \geq \mu(y)$. Since $\mu(y) < 0.5$, then $\mu(0) = \mu(y*y) \geq \min\{\mu(y), \mu(y), 0.5\} = \mu(y)$. Thus $\mu(0 * y) \geq \min\{\mu(0), \mu(y), 0.5\} = \mu(y)$. Hence (1) and hypothesis imply that $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$. Since $\mu(x) < \mu(y)$, then $\mu(x) \geq \mu(x*y)$. Therefore $\mu(x*y) = \mu(x)$. Now let $\mu(x) < 0.5 \leq \mu(y)$. Then similar to above argument $\mu(x * y) \geq \mu(x)$ and $\mu(x) \geq \min\{\mu(x * y), \mu(0 * y), 0.5\}$. Since $\mu(y) \geq 0.5$, then by Theorem 3.22(i), $\mu(0) \geq 0.5$. Thus $\mu(0 * y) \geq \min\{\mu(0), \mu(y), 0.5\} = 0.5$. So by hypothesis we get that $\mu(x) \geq \min\{\mu(x * y), 0.5\}$. Thus $\mu(x) < 0.5$ imply that $\mu(x) \geq \mu(x * y)$. Therefore $\mu(x * y) = \mu(x)$. Let $\mu(x) \geq 0.5$. Then $\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5) = 0.5$.

Theorem 3.24. Let μ be an $(\in, \in \vee q)$-fuzzy subalgebra of X. Then for all $t \in [0, 0.5]$, the nonempty level set $U(\mu; t)$ is a subalgebra of X. Conversely, if the nonempty level set μ is a subalgebra of X, for all $t \in [0, 1]$, then μ is an $(\in, \in \vee q)$-fuzzy subalgebra of X.

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy subalgebra of X. If $t = 0$, then $U(\mu; t)$ is a subalgebra of X. Now let $U(\mu; t) \neq \emptyset$, $0 < t \leq 0.5$ and $x, y \in U(\mu; t)$. Then $\mu(x), \mu(y) \geq t$. Thus by hypothesis we have $\mu(x * y) \geq \min(\mu(x), \mu(y), 0.5) \geq \min(t, 0.5) \geq t$. Therefore $U(\mu; t)$ is a subalgebra of X.

Conversely, let $x, y \in X$. Then we have

$$\mu(x), \mu(y) \geq \min(\mu(x), \mu(y), 0.5) = t_0.$$

Hence $x, y \in U(\mu; t_0)$, for $t_0 \in [0,1]$ and so $x * y \in U(\mu; t_0)$. Therefore $\mu(x * y) \geq t_0 = \min(\mu(x), \mu(y), 0.5)$, i.e μ is an $(\in, \in \vee q)$-fuzzy subalgebra of X.

Theorem 3.25. Let S be a subset of X. The characteristic function χ_S of S is an $(\in, \in \vee q)$-fuzzy subalgebra of X if and only if S is a subalgebra of X.

Proof. Let X_S be an $(\in, \in \vee q)$-fuzzy subalgebra of X and $x, y \in S$. Then $\chi_S(x) = 1 = \chi_S(y)$, and so $x_1 \in \chi_S$ and $y_1 \in \chi_S$. Hence $(x*y)_1 = (x*y)_{\min(1,1)} \in \vee q \chi_S$, which implies that $\chi_S(x * y) > 0$. Thus $x * y \in S$. Therefore S is a subalgebra of X.

Conversely, if S is a subalgebra of X, then χ_S is an $(\in, \in)$-fuzzy subalgebra of X. So by Theorem 3.5 we get that μ is an $(\in, \in \vee q)$-fuzzy subalgebra of X.

Lemma 3.26. Let $f : X \to Y$ be a BG-homomorphism and G be a fuzzy set of Y with membership function μ_G. Then $x_t \alpha \mu_{f^{-1}(G)} \Leftrightarrow f(x)_t \alpha \mu_G$, for all $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$.

Proof. Let $\alpha = \in$. Then

$$x_t \alpha \mu_{f^{-1}(G)} \Leftrightarrow \mu_{f^{-1}(G)}(x) \geq t \Leftrightarrow \mu_G(f(x)) \geq t \Leftrightarrow (f(x))_t \alpha \mu_G.$$

The proof of the other cases is similar to above argument.

Theorem 3.27. Let $f : X \to Y$ be a BG-homomorphism and G be a fuzzy set of Y with membership function μ_G.

(i) If G is an (α, β)-fuzzy subalgebra of Y, then $f^{-1}(G)$ is an (α, β)-fuzzy subalgebra of X,

(ii) Let f be epimorphism. If $f^{-1}(G)$ is an (α, β)-fuzzy subalgebra of X, then G is an (α, β)-fuzzy subalgebra of Y.

Proof. (i) Let $x_t \alpha \mu_{f^{-1}(G)}$ and $y_r \alpha \mu_{f^{-1}(G)}$, for $t, r \in (0, 1]$. Then by Lemma 3.26, we get that $(f(x))_t \alpha \mu_G$ and $(f(y))_r \alpha \mu_G$. Hence by hypothesis $(f(x) * f(y))_{\min(t,r)} \beta \mu_G$. Then $(f(x * y))_{\min(t,r)} \beta \mu_G$ and so $(x * y)_{\min(t,r)} \beta \mu_{f^{-1}(G)}$.

(ii) Let $x, y \in Y$. Then by hypothesis there exist $x', y' \in X$ such that $f(x') = x$ and $f(y') = y$. Assume that $x_t \alpha \mu_G$ and $y_r \alpha \mu_G$, then $(f(x'))_t \alpha \mu_G$ and $(f(y'))_r \alpha \mu_G$. Thus $x'_t \alpha \mu_{f^{-1}(G)}$ and $y'_r \alpha \mu_{f^{-1}(G)}$ and therefore $(x' * y')_{\min(t,r)} \beta \mu_{f^{-1}(G)}$. So

$$(f(x' * y'))_{\min(t,r)} \beta \mu_G \Rightarrow (f(x') * f(y'))_{\min(t,r)} \beta \mu_G \Rightarrow (x * y)_{\min(t,r)} \beta \mu_G.$$

Theorem 3.28. Let $f : X \to Y$ be a BG-homomorphism and H be an $(\in, \in \vee q)$-fuzzy subalgebra of X with membership function μ_H. If μ_H is an f-invariant, then $f(H)$ is an $(\in, \in \vee q)$-fuzzy subalgebra of Y.

Proof. Let y_1 and $y_2 \in Y$. If $f^{-1}(y_1)$ or $f^{-1}(y_2) = \emptyset$, then $\mu_{f(H)}(y_1 * y_2) \geq \min(\mu_{f(H)}(y_1),\ \mu_{f(H)}(y_2), 0.5)$. Now let $f^{-1}(y_1)$ and $f^{-1}(y_2) \neq \emptyset$. Then there exist $x_1,\ x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Thus by hypothesis we have

$$\begin{aligned}\mu_{f(H)}(y_1 * y_2) &= \sup_{t \in f^{-1}(y_1 * y_2)} \mu_H(t) \\ &= \sup_{t \in f^{-1}(f(x_1 * x_2))} \mu_H(t)\end{aligned}$$

$$= \mu_H(x_1 * x_2) \qquad \text{since } \mu_H \text{ is an } f\text{-invariant}$$

$$\geq \min(\mu_H(x_1),\ \mu_H(x_2),\ 0.5)$$

$$= \min(\sup_{t\in f^{-1}(y_1)} \mu_H(t), \sup_{t\in f^{-1}(y_2)} \mu_H(t), 0.5)$$

$$= \min(\mu_{f(H)}(y_1),\ \mu_{f(H)}(y_2),\ 0.5).$$

So by Theorem 3.21, $f(H)$ is an $(\in, \in\vee q)$-fuzzy subalgebra of Y.

Lemma 3.29. Let $f : X \to Y$ be a BG-homomorphism.

(i) If S is a subalgebra of X, then $f(S)$ is a subalgebra of Y;

(ii) If S' is a subalgebra of Y, then $f^{-1}(S')$ is a subalgebra of X.

Proof. The proof is easy.

Theorem 3.30. Let $f : X \to Y$ be a BG-homomorphism . If H is a non-zero (q, q)-fuzzy subalgebra of X with membership function μ_H, then $f(H)$ is a non-zero (q, q)-fuzzy subalgebra of Y.

Proof. Let H be a non-zero (q, q)-fuzzy subalgebra of X. Then by Theorem 3.10, we have $\mu_H(x) = \begin{cases} \mu_H(0) & \text{if } x \in X_0 \\ 0 & \text{otherwise} \end{cases}$. Now we show that $\mu_{f(H)}(y) = \begin{cases} \mu_H(0) & \text{if } y \in f(X_0) \\ 0 & \text{otherwise} \end{cases}$. Let $y \in Y$. If $y \in f(X_0)$, then there exist $x \in X_0$ such that $f(x) = y$. Thus $\mu_{f(H)}(y) = \sup_{t\in f^{-1}(y)} \mu_H(t) = \mu_H(0)$. If $y \notin f(X_0)$, then it is clear that $\mu_{f(H)}(y) = 0$. Since X_0 is subalgebra of X, then $f(X_0)$ is a subalgebra of Y. Therefore by Theorem 3.11, $f(H)$ is a non-zero (q, q)-fuzzy subalgebra of Y.

Theorem 3.31. Let $f : X \to Y$ be a BG-homomorphism . If H is an (α, β)-fuzzy subalgebra of X with membership function μ_H, then $f(H)$ is an (α, β)-fuzzy subalgebra of Y, where (α, β) is one of the following form

(i) $(\in, q)$,	(ii) $(\in, \in\wedge q)$,
(iii) $(q, \in)$,	(iv) $(q, \in\wedge q)$,
(v) $(\in\vee q, q)$,	(vi) $(\in\vee q, \in\wedge q)$,
(vii) $(\in\vee q, \in)$,	(viii) $(q, \in\vee q)$.

Proof. The proof is similar to the proof of Theorem 3.30, by using of Theorems 3.15 and 3.18.

Theorem 3.32. Let $f : X \to Y$ be a BG-homomorphism and H be an $(\in, \in)$-fuzzy subalgebra of X with membership function μ_H. If μ_H is an f-invariant, then $f(H)$ is an $(\in, \in)$-fuzzy subalgebra of Y.

Proof. Let $z_t \in \mu_{f(H)}$ and $y_r \in \mu_{f(H)}$, where $t, r \in (0, 1]$. Then $\mu_{f(H)}(z) \geq t$ and $\mu_{f(H)}(y) \geq r$. Thus $f^{-1}(z), f^{-1}(y) \neq \emptyset$ imply that there exists $x_1, x_2 \in X$ such that $f(x_1) = z$ and $f(x_2) = y$. since μ_H is an f-invariant, then $\mu_{f(H)}(z) \geq t$ and $\mu_{f(H)}(y) \geq r$ imply that $\mu_H(x_1) \geq t$ and $\mu_H(x_2) \geq r$. So by hypothesis we have

$$\begin{aligned}\mu_{f(H)}(z*y) &= \sup_{t\in f^{-1}(z*y)} \mu_H(t)\\ &= \sup_{t\in f^{-1}(f(x_1*x_2))} \mu_H(t)\\ &= \mu_H(x_1*x_2)\\ &\geq \min(t,r).\end{aligned}$$

Therefore $(z*y)_{\min(t,r)} \in \mu_{f(H)}$, i.e $f(H)$ is an $(\in,\in)$-fuzzy subalgebra of Y.

Theorem 3.33. Let $\{\mu_i \mid i\in\Lambda\}$ be a family of $(\in,\in\vee q)$-fuzzy subalgebra of X. Then $\mu := \bigcap_{i\in\Lambda}\mu_i$ is an $(\in,\in\vee q)$-fuzzy subalgebra of X.

Proof. By Theorem 3.21 we have, for all $i\in\Lambda$

$$\mu_i(x*y) \geq \min(\mu_i(x),\mu_i(y),0.5).$$

Therefore
$$\begin{aligned}\mu(x*y) = \inf_{i\in\Lambda}\mu_i(x*y) &\geq \inf_{i\in\Lambda}\min(\mu_i(x),\mu_i(y),0.5)\\ &= \min(\inf_{i\in\Lambda}\mu_i(x),\inf_{i\in\Lambda}\mu_i(y),0.5)\\ &= \min(\mu(x),\mu(y),0.5).\end{aligned}$$

Therefore by Theorem 3.21, μ is an $(\in,\in\vee q)$-fuzzy subalgebra.

Theorem 3.34. Let $\{\mu_i \mid i\in\Lambda\}$ be a family of $(\in,\in)$-fuzzy subalgebra of X. Then $\mu := \bigcap_{i\in\Lambda}\mu_i$ is an $(\in,\in)$-fuzzy subalgebra of X.

Proof. Let $x_t\in\mu$ and $y_r\in\mu$, $t,r\in(0,1]$. Then $\mu(x)\geq t$ and $\mu(y)\geq r$. Thus for all $i\in\Lambda$, $\mu_i(x)\geq t$ and $\mu_i(y)\geq r$ imply that $\mu_i(x*y)\geq\min(t,r)$. Therefore $\mu(x*y)\geq\min(t,r)$ i.e $(x*y)_{\min(t,r)}\in\mu$.

Theorem 3.35. Let $\{\mu_i \mid i\in\Lambda\}$ be a family of (α,β)-fuzzy subalgebra of X. Then $\mu := \bigcap_{i\in\Lambda}\mu_i$ is an (α,β)-fuzzy subalgebra of X, where (α,β) is one of the following form

(i) $(\in,q)$, (ii) $(\in,\wedge q)$,

(iii) $(q,\in)$, (iv) $(q,\in\wedge q)$,

(v) $(\in\vee q,q)$, (vi) $(\in\vee q,\in\wedge q)$,

(vii) $(\in\vee q,\in)$, (viii) $(q,\in\vee q)$,

(ix) (q,q).

Proof. We prove theorem for (q,q)-fuzzy subalgebra. The proof of the other cases is similar, by using Theorems 3.15 and 3.18.

If there exists $i\in\Lambda$ such that $\mu_i=0$, then $\mu=0$. So μ is a (q,q)-fuzzy subalgebra. Let $\mu_i\neq 0$ for all $i\in\Lambda$. Then by Theorem 3.10 we have $\mu_i(x)=\begin{cases}\mu_i(0) & \text{if } x\in X_0^i\\ 0 & \text{otherwise}\end{cases}$, for all

$i \in \Lambda$. So it is clear that $\mu(x) = \begin{cases} \mu(0) & \text{if } x \in \bigcap_{i\in\Lambda} X_0^i \\ 0 & \text{otherwise} \end{cases}$. Since $\bigcap_{i\in\Lambda} X_0^i$ is a subalgebra of X, then by Theorem 3.11 μ is a (q, q)-fuzzy subalgebra of X.

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On the Smarandache totient function and the Smarandache power sequence

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Abstract For any positive integer n, let $SP(n)$ denotes the Smarandache power sequence. And for any Smarandache sequence $a(n)$, the Smarandache totient function $St(n)$ is defined as $\varphi(a(n))$, where $\varphi(n)$ is the Euler totient function. The main purpose of this paper is using the elementary and analytic method to study the convergence of the function $\frac{S_1}{S_2}$, where $S_1 = \sum_{k=1}^{n}\left(\frac{1}{St(k)}\right)^2$, $S_2 = \left(\sum_{k=1}^{n}\frac{1}{St(k)}\right)^2$, and give an interesting limit Theorem.

Keywords Smarandache power function, Smarandache totient function, convergence.

§1. Introduction and results

For any positive integer n, the Smarandache power function $SP(n)$ is defined as the smallest positive integer m such that $n \mid m^m$, where m and n have the same prime divisors. That is,

$$SP(n) = \min\left\{m:\ n \mid m^m,\ m \in N,\ \prod_{p|n} p = \prod_{p|m} p\right\}.$$

For example, the first few values of $SP(n)$ are: $SP(1) = 1$, $SP(2) = 2$, $SP(3) = 3$, $SP(4) = 2$, $SP(5) = 5$, $SP(6) = 6$, $SP(7) = 7$, $SP(8) = 4$, $SP(9) = 3$, $SP(10) = 10$, $SP(11) = 11$, $SP(12) = 6$, $SP(13) = 13$, $SP(14) = 14$, $SP(15) = 15$, $\cdots$. In reference [1], Professor F.Smarandache asked us to study the properties of $SP(n)$. It is clear that $SP(n)$ is not a multiplicative function. For example, $SP(8) = 4$, $SP(3) = 3$, $SP(24) = 6 \neq SP(3) \times SP(8)$. But for most n, we have $SP(n) = \prod_{p|n} p$, where $\prod_{p|n}$ denotes the product over all different prime divisors of n. If $n = p^\alpha$, $k \cdot p^k + 1 \leq \alpha \leq (k+1)p^{k+1}$, then we have $SP(n) = p^{k+1}$, where $0 \leq k \leq \alpha - 1$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, for all α_i $(i = 1, 2, \cdots, r)$, if $\alpha_i \leq p_i$, then $SP(n) = \prod_{p|n} p$.

About other properties of the function $SP(n)$, many authors had studied it, and gave some interesting conclusions. For example, in reference [4], Zhefeng Xu had studied the mean value properties of $SP(n)$, and obtained a sharper asymptotic formula:

$$\sum_{n \leq x} SP(n) = \frac{1}{2}x^2 \prod_p \left(1 - \frac{1}{p(p+1)}\right) + O\left(x^{\frac{3}{2}} + \epsilon\right),$$

where ϵ denotes any fixed positive number, and $\prod_p$ denotes the product over all primes.

On the other hand, similar to the famous Euler totient function $\varphi(n)$, Professor F.Russo defined a new arithmetical function called the Smarandache totient function $St(n) = \varphi(a(n))$, where $a(n)$ is any Smarandache sequence. Then he asked us to study the properties of these functions. At the same time, he proposed the following:

Conjecture. For the Smarandache power sequence $SP(k)$, $\frac{S_1}{S_2}$ converges to zero as $n \to \infty$, where $S_1 = \sum_{k=1}^{n} \left(\frac{1}{St(k)}\right)^2$, $S_2 = \left(\sum_{k=1}^{n} \frac{1}{St(k)}\right)^2$.

In this paper, we shall use the elementary and analytic methods to study this problem, and prove that the conjecture is correct. That is, we shall prove the following:

Theorem. For the Smarandache power function $SP(k)$, we have $\lim_{n\to\infty} \frac{S_1}{S_2} = 0$, where $S_1 = \sum_{k=1}^{n} \left(\frac{1}{\varphi(SP(k))}\right)^2$, $S_2 = \left(\sum_{k=1}^{n} \frac{1}{\varphi(SP(k))}\right)^2$.

§2. Some lemmas

To complete the proof of the theorem, we need the following two simple Lemmas:

Lemma 1. For any given real number $\epsilon > 0$, there exists a positive integer $N(\epsilon)$, such that for all $n \geq N(\epsilon)$, we have $\varphi(n) \geq (1-\epsilon)\frac{c \cdot n}{\ln \ln n}$, where c is a constant.

Proof. See reference [5].

Lemma 2. For the Euler totient fuction $\varphi(n)$, we have the asymptotic formula

$$\sum_{k \leq n} \frac{1}{\varphi(k)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \ln n + A + O\left(\frac{\ln n}{n}\right),$$

where $A = \gamma \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n\varphi(n)} - \sum_{n=1}^{\infty} \frac{\mu^2(n) \ln n}{n\varphi(n)}$ is a constant.

Proof. See reference [6].

§3. Proof of the theorem

In this section, we shall prove our Theorem.

We separate all integer k in the interval $[1, n]$ into two subsets A and B as follows: A : the set of all square-free integers. B : the set of other positive integers k such that $k \in [1, n] \backslash A$. So we have

$$\sum_{k \leq n} \frac{1}{(\varphi(SP(k)))^2} = \sum_{k \in A} \frac{1}{(\varphi(SP(k)))^2} + \sum_{k \in B} \frac{1}{(\varphi(SP(k)))^2}.$$

From the definition of the subset A, we may get

$$\sum_{k \in A} \frac{1}{(\varphi(SP(k)))^2} = \sum_{k \in A} \frac{1}{k^2 \prod_{p|k} \left(1 - \frac{1}{p}\right)^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2 \prod_{p|k} \left(1 - \frac{1}{p}\right)^2} \ll 1.$$

By Lemma 1, we can easily get $\dfrac{k}{\varphi(k)} = O(\ln\ln k)$. Note that $\sum\limits_{k\leq n}\dfrac{\mu^2(k)}{k^2} = O(1)$. And if $k\in B$, then we can write k as $k = l\cdot m$, where l is a square-free integer and m is a square-full integer. Let S denote $\sum\limits_{k\in B}\dfrac{1}{(\varphi(SP(k)))^2}$, then from the properties of $SP(k)$ and $\varphi(k)$ we have

$$S \leq \sum_{lm\leq n}\frac{1}{l^2\prod\limits_{p|m}p^2\prod\limits_{p|lm}\left(1-\frac{1}{p}\right)^2} = \sum_{m\leq n}\frac{1}{\prod\limits_{p|m}p^2}\sum_{l\leq\frac{n}{m}}\frac{\mu^2(l)}{l^2}\cdot\frac{l^2m^2}{\varphi^2(lm)} = O\left((\ln\ln n)^2\sum_{m\leq n}\frac{1}{\prod\limits_{p|m}p^2}\right).$$

Let $U(k) = \prod\limits_{p|k}p$, then $\sum\limits_{m\leq n}\dfrac{1}{\prod\limits_{p|m}p^2} = \sum\limits_{k\leq n}\dfrac{a(k)}{U^2(k)}$, where m is a square-full integer and the arithmetical function $a(k)$ is defined as follows:

$$a(k) = \begin{cases} 1, & \text{if } k \text{ is a square-full integer;}\\ 0, & \text{otherwise.}\end{cases}$$

Note that $\dfrac{a(k)}{U^2(k)}$ is a multiplicative function. According to the Euler product formula (see reference [3] and [5]), we have

$$A(s) = \sum_{k=1}^{\infty}\frac{a(k)}{U^2(k)k^s} = \prod_p\left(1+\frac{1}{p^{2+s}(p^s-1)}\right).$$

From the Perron formulas [5], for $b = 1+\dfrac{1}{\ln n}$, $T\geq 1$, we have

$$\sum_{k\leq n}\frac{a(k)}{U^2(k)} = \frac{1}{2\pi i}\int_{b-iT}^{b+iT}A(s)\frac{n^s}{s}ds + O\left(\frac{n^b\zeta(b)}{T}\right) + O\left(n\min\left(1,\frac{\ln n}{T}\right)\right) + \frac{a(n)}{2U^2(n)}.$$

Taking $T = n$, we can get the estimate

$$O\left(\frac{n^b\zeta(b)}{T}\right) + O\left(n\min\left(1,\frac{\ln n}{T}\right)\right) + \frac{a(n)}{2U^2(n)} = O(\ln n).$$

Because the function $A(s)\dfrac{n^s}{s}$ is analytic in $Re\ s > 0$, taking $c = \dfrac{1}{\ln n}$, then we have

$$\frac{1}{2\pi i}\left(\int_{b-iT}^{b+iT}A(s)\frac{n^s}{s}ds + \int_{c-iT}^{b-iT}A(s)\frac{n^s}{s}ds + \int_{b+iT}^{c+iT}A(s)\frac{n^s}{s}ds + \int_{c+iT}^{c-iT}A(s)\frac{n^s}{s}ds\right) = 0.$$

Note that $\displaystyle\int_{c-iT}^{c+iT}A(s)\frac{n^s}{s}ds = O\left(\int_{-T}^{T}\frac{dy}{\sqrt{c^2+y^2}}\right) = O(\ln n)$ and $\displaystyle\int_{c-iT}^{b-iT}A(s)\frac{n^s}{s}ds = O\left(\int_c^b\frac{n^\sigma}{T}d\sigma\right) = O\left(\frac{1}{\ln n}\right)$. Similarly, $\displaystyle\int_{c+iT}^{b+iT}A(s)\frac{n^s}{s}ds = O\left(\frac{1}{\ln n}\right)$. Hence, $\displaystyle\sum_{k\leq n}\frac{a(k)}{U^2(k)} = O(\ln n)$.

So

$$\sum_{k\leq n}\frac{1}{(\varphi(SP(k)))^2}=O(\ln n\cdot(\ln\ln n)^2). \tag{1}$$

Now we come to estimate $\sum_{k\leq n}\frac{1}{\varphi(SP(k))}$, from the definition of $SP(n)$, we may immediately get that $SP(n)\leq n$. Let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ denotes the factorization of n into prime powers, then $SP(n)=p_1^{\beta_1}p_2^{\beta_2}\cdots p_s^{\beta_s}$, where $\beta_i\geq 1$. Therefore, we can get that $p_1^{\beta_1-1}(p_1-1)p_2^{\beta_2-1}(p_2-1)\cdots p_s^{\beta_s-1}(p_s-1)\leq p_1^{\alpha_1-1}(p_1-1)p_2^{\alpha_2-1}(p_2-1)\cdots p_s^{\alpha_s-1}(p_s-1)$, thus $\varphi(p_1^{\beta_1}p_2^{\beta_2}\cdots p_s^{\beta_s})\leq \varphi(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s})$. That is, $\varphi(SP(n))\leq\varphi(n)$, according to Lemma 2, we can easily get

$$\sum_{k\leq n}\frac{1}{\varphi(SP(k))}\geq\sum_{k\leq n}\frac{1}{\varphi(k)}=\frac{\zeta(2)\zeta(3)}{\zeta(6)}\ln n+A+O\left(\frac{\ln n}{n}\right). \tag{2}$$

Combining (1) and (2), we obtain

$$0\leq\frac{\sum\limits_{k=1}^{n}\left(\frac{1}{\varphi(SP(k))}\right)^2}{\left(\sum\limits_{k=1}^{n}\frac{1}{\varphi(SP(k))}\right)^2}\leq\frac{O\left(\ln n\cdot(\ln\ln n)^2\right)}{\left(\frac{\zeta(2)\zeta(3)}{\zeta(6)}\ln n+A+O\left(\frac{\ln n}{n}\right)\right)^2}\longrightarrow 0,\quad \text{as } n\to\infty.$$

This completes the proof of our Theorem.

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A new additive function and the F. Smarandache function

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Abstract For any positive integer n, we define the arithmetical function $F(n)$ as $F(1) = 0$. If $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime power factorization of n, then $F(n) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_k p_k$. Let $S(n)$ be the Smarandache function. The main purpose of this paper is using the elementary method and the prime distribution theory to study the mean value properties of $(F(n) - S(n))^2$, and give a sharper asymptotic formula for it.

Keywords Additive function, Smarandache function, Mean square value, Elementary method, Asymptotic formula.

§1. Introduction and result

Let $f(n)$ be an arithmetical function, we call $f(n)$ as an additive function, if for any positive integers m, n with $(m,\ n) = 1$, we have $f(mn) = f(m) + f(n)$. We call $f(n)$ as a complete additive function, if for any positive integers r and s, $f(rs) = f(r) + f(s)$. In elementary number theory, there are many arithmetical functions satisfying the additive properties. For example, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the prime power factorization of n, then function $\Omega(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ and logarithmic function $f(n) = \ln n$ are two complete additive functions, $\omega(n) = k$ is an additive function, but not a complete additive function. About the properties of the additive functions, one can find them in references [1], [2] and [5].

In this paper, we define a new additive function $F(n)$ as follows: $F(1) = 0$; If $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the prime power factorization of n, then $F(n) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_k p_k$. It is clear that this function is a complete additive function. In fact if $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, then we have $mn = p_1^{\alpha_1+\beta_1} p_2^{\alpha_2+\beta_2} \cdots p_k^{\alpha_k+\beta_k}$. Therefore, $F(mn) = (\alpha_1 + \beta_1)p_1 + (\alpha_2 + \beta_2)p_2 + \cdots + (\alpha_k + \beta_k)p_k = F(m) + F(n)$. So $F(n)$ is a complete additive function. Now we let $S(n)$ be the Smarandache function. That is, $S(n)$ denotes the smallest positive integer m such that n divide $m!$, or $S(n) = \min\{m :\ n \mid m!\}$. About the properties of $S(n)$, many authors had studied it, and obtained a series results, see references [7], [8] and [9]. The main purpose of this paper is using the elementary method and the prime distribution theory to study the mean value properties of $(F(n) - S(n))^2$, and give a sharper asymptotic formula for it. That is, we shall prove the following:

Theorem. Let N be any fixed positive integer. Then for any real number $x > 1$, we

have the asymptotic formula

$$\sum_{n\leq x}(F(n)-S(n))^2=\sum_{i=1}^{N}c_i\cdot\frac{x^2}{\ln^{i+1}x}+O\left(\frac{x^2}{\ln^{N+2}\sqrt{x}}\right),$$

where c_i $(i=1,\ 2,\ \cdots,\ N)$ are computable constants, and $c_1=\frac{\pi^2}{6}$.

§2. Proof of the theorem

In this section, we use the elementary method and the prime distribution theory to complete the proof of the theorem. We using the idea in reference [4]. First we define four sets A, B, C, D as follows: $A=\{n,\ n\in N$, n has only one prime divisor p such that $p\mid n$ and $p^2\nmid n$, $p>n^{\frac{1}{3}}\}$; $B=\{n,\ n\in N$, n has only one prime divisor p such that $p^2\mid n$ and $p>n^{\frac{1}{3}}\}$; $C=\{n,\ n\in N$, n has two deferent prime divisors p_1 and p_2 such that $p_1p_2\mid n$, $p_2>p_1>n^{\frac{1}{3}}\}$; $D=\{n,\ n\in N$, any prime divisor p of n satisfying $p\leq n^{\frac{1}{3}}\}$, where N denotes the set of all positive integers. It is clear that from the definitions of A, B, C and D we have

$$\begin{aligned}\sum_{n\leq x}(F(n)-S(n))^2 &= \sum_{\substack{n\leq x\\ n\in A}}(F(n)-S(n))^2+\sum_{\substack{n\leq x\\ n\in B}}(F(n)-S(n))^2\\ &\quad+\sum_{\substack{n\leq x\\ n\in C}}(F(n)-S(n))^2+\sum_{\substack{n\leq x\\ n\in D}}(F(n)-S(n))^2\\ &\equiv W_1+W_2+W_3+W_4. \qquad (1)\end{aligned}$$

Now we estimate W_1, W_2, W_3 and W_4 in (1) respectively. Note that $F(n)$ is a complete additive function, and if $n\in A$ with $n=pk$, then $S(n)=S(p)=p$, and any prime divisor q of k satisfying $q\leq n^{\frac{1}{3}}$, so $F(k)\leq n^{\frac{1}{3}}\ln n$. From the Prime Theorem (See Chapter 3, Theorem 2 of [3]) we know that

$$\pi(x)=\sum_{p\leq x}1=\sum_{i=1}^{k}c_i\cdot\frac{x}{\ln^i x}+O\left(\frac{x}{\ln^{k+1}x}\right), \qquad (2)$$

where c_i $(i=1,\ 2,\cdots,\ k)$ are computable constants, and $c_1=1$. By these we have the estimate:

$$\begin{aligned}W_1 &= \sum_{\substack{n\leq x\\ n\in A}}(F(n)-S(n))^2=\sum_{\substack{pk\leq x\\ (pk)\in A}}(F(pk)-p)^2\\ &= \sum_{\substack{pk\leq x\\ (pk)\in A}}F^2(k)\ll\sum_{k\leq\sqrt{x}}\sum_{k<p\leq\frac{x}{k}}(pk)^{\frac{2}{3}}\ln^2(pk)\leq(\ln x)^2\sum_{k\leq\sqrt{x}}k^{\frac{2}{3}}\sum_{k<p\leq\frac{x}{k}}p^{\frac{2}{3}}\\ &\ll (\ln x)^2\sum_{k\leq\sqrt{x}}k^{\frac{2}{3}}\left(\frac{x}{k}\right)^{\frac{5}{3}}\frac{1}{\ln\frac{x}{k}}\ll x^{\frac{5}{3}}\ln^2 x. \qquad (3)\end{aligned}$$

If $n \in B$, then $n = p^2 k$, and note that $S(n) = S(p^2) = 2p$, we have the estimate

$$\begin{aligned} W_2 &= \sum_{\substack{n\le x\\ n\in B}} (F(n)-S(n))^2 = \sum_{\substack{p^2k\le x\\ p>k}} \left(F(p^2k)-2p\right)^2 \\ &= \sum_{k\le x^{\frac{1}{3}}} \sum_{k<p\le\sqrt{\frac{x}{k}}} F^2(k) \ll \sum_{k\le x^{\frac{1}{3}}} \sum_{k<p\le\sqrt{\frac{x}{k}}} k^2 \\ &\ll \sum_{k\le x^{\frac{1}{3}}} \frac{k^2\cdot x^{\frac{1}{2}}}{k^{\frac{1}{2}}\ln x} \ll \frac{x^{\frac{4}{3}}}{\ln x}. \end{aligned} \tag{4}$$

If $n \in D$, then $F(n) \le n^{\frac{1}{3}} \ln n$ and $S(n) \le n^{\frac{1}{3}} \ln n$, so we have

$$W_4 = \sum_{\substack{n\le x\\ n\in D}} (F(n)-S(n))^2 \ll \sum_{n\le x} n^{\frac{2}{3}} \ln^2 n \ll x^{\frac{5}{3}} \ln^2 x. \tag{5}$$

Finally, we estimate main term W_3. Note that $n \in C$, $n = p_1p_2k$, $p_2 > p_1 > n^{\frac{1}{3}} > k$. If $k < p_1 < n^{\frac{1}{3}}$, then in this case, the estimate is exact same as in the estimate of W_1. If $k < p_1 < p_2 < n^{\frac{1}{3}}$, in this case, the estimate is exact same as in the estimate of W_4. So by (2) we have

$$\begin{aligned} W_3 &= \sum_{\substack{n\le x\\ n\in C}} (F(n)-S(n))^2 = \sum_{\substack{p_1p_2k\le x\\ p_2>p_1>k}} (F(p_1p_2k)-p_2)^2 + O\left(x^{\frac{5}{3}}\ln^2 x\right) \\ &= \sum_{k\le x^{\frac{1}{3}}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} \sum_{p_2\le\frac{x}{p_1k}} \left(F^2(k)+2p_1F(k)+p_1^2\right) + O\left(x^{\frac{5}{3}}\ln^2 x\right) \\ &= \sum_{k\le x^{\frac{1}{3}}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} \sum_{p_1<p_2\le\frac{x}{p_1k}} p_1^2 + O\left(\sum_{k\le x^{\frac{1}{3}}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} \sum_{p_1<p_2\le\frac{x}{p_1k}} kp_1\right) + O\left(x^{\frac{5}{3}}\ln^2 x\right) \\ &= \sum_{k\le x^{\frac{1}{3}}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} p_1^2 \left(\sum_{i=1}^{N} c_i\cdot\frac{x}{p_1k\ln^i\frac{x}{p_1k}} + O\left(\frac{x}{p_1k\ln^{N+1}x}\right)\right) + O\left(x^{\frac{5}{3}}\ln^2 x\right) \\ &\quad - \sum_{k\le x^{\frac{1}{3}}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} p_1^2 \sum_{p_2\le p_1} 1 + O\left(\sum_{k\le x^{\frac{1}{3}}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} \sum_{p_1<p_2\le\frac{x}{p_1k}} kp_1\right). \end{aligned} \tag{6}$$

Note that $\zeta(2) = \frac{\pi^2}{6}$, from the Abel's identity (See Theorem 4.2 of [6]) and (2) we have

$$\begin{aligned} &\sum_{k\le x^{\frac{1}{3}}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} p_1^2 \sum_{p\le p_1} 1 = \sum_{k\le x^{\frac{1}{3}}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} p_1^2 \left[\sum_{i=1}^{N} \frac{c_i\cdot p_1}{\ln^i p_1} + O\left(\frac{p_1}{\ln^{N+1} p_1}\right)\right] \\ &= \sum_{i=1}^{N} \sum_{k\le x^{\frac{1}{3}}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} \frac{c_i\cdot p_1^3}{\ln^i p_1} + O\left(\sum_{k\le x^{\frac{1}{3}}} \sum_{k<p_1\le\sqrt{\frac{x}{k}}} \frac{p_1^3}{\ln^{N+1} p_1}\right) \\ &= \sum_{i=1}^{N} \frac{d_i\cdot x^2}{\ln^{i+1} x} + O\left(\frac{2^N\cdot x^2}{\ln^{N+2} x}\right), \end{aligned} \tag{7}$$

where d_i $(i = 1,\ 2,\ \cdots,\ N)$ are computable constants, and $d_1 = \frac{\pi^2}{6}$.

$$\sum_{k\le x^{\frac{1}{3}}}\ \sum_{k<p_1\le\sqrt{\frac{x}{k}}}\ \sum_{p_1<p_2\le\frac{x}{p_1k}} kp_1 \ll \sum_{k\le x^{\frac{1}{3}}} k \sum_{p_1\le\sqrt{\frac{x}{k}}} p_1\cdot\frac{x}{p_1k\ln x} \ll \sum_{k\le x^{\frac{1}{3}}}\frac{x^{\frac{3}{2}}}{\sqrt{k}\ln^2 x} \ll \frac{x^{\frac{5}{3}}}{\ln^2 x}. \tag{8}$$

$$\sum_{k\le x^{\frac{1}{3}}}\ \sum_{k<p_1\le\sqrt{\frac{x}{k}}}\frac{p_1x}{k\ln^{N+1}x} \ll \sum_{k\le x^{\frac{1}{3}}}\frac{x^2}{k^2\ln^{N+2}x} \ll \frac{x^2}{\ln^{N+2}x}. \tag{9}$$

From the Abel's identity and (2) we also have the estimate

$$\begin{aligned}
&\sum_{k\le x^{\frac{1}{3}}}\ \sum_{k<p_1\le\sqrt{\frac{x}{k}}} p_1^2\frac{x}{p_1k\ln\frac{x}{p_1k}} = \sum_{k\le x^{\frac{1}{3}}}\frac{1}{k}\sum_{k<p_1\le\sqrt{\frac{x}{k}}}\frac{xp_1}{\ln\frac{x}{kp_1}}\\
=\ &\sum_{i=1}^{N} b_i\cdot\frac{x^2}{\ln^{i+1}x} + O\left(\frac{x^2}{\ln^{N+1}x}\right),
\end{aligned} \tag{10}$$

where b_i $(i = 1,\ 2,\ \cdots,\ N)$ are computable constants, and $b_1 = \frac{\pi^2}{3}$.

Now combining (1), (3), (4), (5), (6), (7), (8)and(9) we may immediately deduce the asymptotic formula:

$$\sum_{n\le x}(F(n)-S(n))^2 = \sum_{i=1}^{N} a_i\cdot\frac{x^2}{\ln^{i+1}x} + O\left(\frac{x^2}{\ln^{N+2}\sqrt{x}}\right),$$

where a_i $(i = 1,\ 2,\ \cdots,\ N)$ are computable constants, and $a_1 = b_1 - d_1 = \frac{\pi^2}{6}$.

This completes the proof of Theorem.

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